Journal of Statistical Physics, Vol. 124, Nos. 2-4, July/August 2006 (© 2006)
DOI: 10.1007/s10955-005-8008-8

On a Taylor-Couette Type Bifurcation for the Stationary Nonlinear Boltzmann Equation

L. Arkeryd[1] and A. Nouri[2]

Received May 12, 2005; accepted July 26, 2005; Published Online: January 20, 2006

This paper studies the stationary nonlinear Boltzmann equation for hard forces, in a Taylor-Couette setting between two coaxial, rotating cylinders with given indata of Maxwellian type on the cylinders. A priori L^q-estimates are obtained, and used to prove a Taylor type bifurcation with isolated solutions and a hydrodynamic limit control, based on asymptotic expansions together with a rest term correction. The positivity of such solutions is also considered.

KEY WORDS: Stationary Boltzmann equation; Couette problem; Taylor-Couette bifurcation; Positive solutions; 82D05.

1. INTRODUCTION

For the stationary nonlinear Boltzmann equation (cf ref.[3]) away from equilibrium, weak compactness arguments may be employed to prove existence–in the stationary case usually involving entropy dissipation control for the sharpest results. On the other hand, such an approach is too general to provide information about uniqueness, isolated solutions, or details about fluid limits. That type of results, has so far had to be based on the asymptotic methods initiated by Grad,[8,9] Kogan[11] and Guiraud[10] in the 1960s and 1970s. For a short review of the development, see our previous paper,[1] which is mainly concerned with the existence of multiple isolated solutions and their fluid limits. Those results were obtained with the help of some technical observations, which are further developed in the present study dealing with a Taylor-Couette bifurcation for the stationary nonlinear Boltzmann equation with Maxwellian ingoing boundary values between two coaxial cylinders

[1] Department of Mathematics, Chalmers Institute of Technology, Gothenburg, Sweden; e-mail: arkeryd@math.chalmers.se
[2] CMI, Université d'Aix-Marseille I, Marseille, France.

A and B and including the small mean free path asymptotics in a neighbourhood of the bifurcation point. The paper was inspired by a treatment of the same set-up by Sone and Doi[14] from a numerical and asymptotic perspective, to which we also refer for a discussion of more applied aspects.

The boundary values are assumed to be axially and circumferentially uniform in the space variables, and the solutions circumferentially uniform and periodic in the z variable. Then, with (r, θ, z) and (v_r, v_θ, v_z) respectively denoting the spatial cylindrical coordinates and the corresponding velocity coordinates, a distribution function may be written as $f = f(r, z, v_r, v_\theta, v_z)$, and the Boltzmann equation becomes

$$v_r \frac{\partial f}{\partial r} + v_z \frac{\partial f}{\partial z} + \frac{1}{r} N f = \frac{1}{\epsilon} Q(f, f), \qquad r \in (r_A, r_B), (v_r, v_\theta, v_z) \in \mathbb{R}^3. \quad (1.1)$$

With respect to the axial variable z the functions f are either constant or periodic (and for simplicity with period $r_B - r_A$ in a main case discussed). The Maxwellian ingoing, axially uniform boundary data under study are

$$\begin{aligned} \gamma^+ f(r_A, z, v) &= \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}(v_r^2 + (v_\theta - \epsilon u_{\theta A1})^2 + v_z^2)}, \quad v_r > 0, \\ \gamma^+ f(r_B, z, v) &= \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2} v^2}, \quad v_r < 0. \end{aligned} \quad (1.2)$$

Here

$$\begin{aligned} Nf &:= v_\theta^2 \frac{\partial f}{\partial v_r} - v_\theta v_r \frac{\partial f}{\partial v_\theta}, \\ Q(f, f)(v) &:= \int_{I\!R^3 \times S^2} B(v - v_*, \omega)(f(v') f(v'_*) - f(v) f(v_*)) dv_* d\omega. \end{aligned} \quad (1.3)$$

The kernel $B = |v - v_*|^\beta b(\theta),\ b \in L^1_+(S^2),\ 0 \leq \beta \leq 1$, is of hard force type and assumed to belong to the Grad class, that is with its terms suitably majorized by the corresponding ones for the hard sphere model (cf ref.[12]). The case $\beta = 0$ corresponds to Maxwellian molecules and $\beta = 1$ to hard spheres. Consider functions with the symmetry

$$f(r, z, v_r, v_\theta, v_z) = f(r, -z, v_r, v_\theta, -v_z),\ r \in [r_A, r_B],\ z \in I\!R, v \in I\!R^3.$$

Take the radii as $r_A = 1, r_B > 1$, and let ϵ denote the Knudsen number. The given rotational velocities of the inner and outer cylinders are in the same direction with $u_{\theta A} = \epsilon u_{\theta A1}$ and $u_{\theta B} = 0$ respectively. The non-dimensional perturbed relative temperature and saturated pressure are $\tau_B = P_B = 0$.

Denote by $\| \, . \, \|_2$ the usual Lebesgue L^2-norm – with the weight M added in case of velocity space – and set for $1 \leq q \leq +\infty$,

$$\tilde{L}^q := \left\{ f; | \, f \, |_q := \left(\int M(v) \left(\int | \, f(x,v) \, |^q \, dx \right)^{\frac{2}{q}} dv \right)^{\frac{1}{2}} < +\infty \right\},$$

where $M = (2\pi)^{-\frac{3}{2}} \exp\left(-\frac{v^2}{2}\right)$. Write $R = f_{\text{rest}} = P_0 f_{\text{rest}} + (I - P_0) f_{\text{rest}} = R_{\|} + R_{\perp}$, where P_0 is the orthogonal projection on the hydrodynamic part, and

$$f = M(1 + \varphi + \epsilon^{j_0} f_{rest}) \quad \text{with } \varphi = \sum_1^{j_1} \epsilon^j \Phi^j. \tag{1.4}$$

Here $\sum_1^{j_1} \epsilon^j \Phi^j$ is an asymptotic expansion with the boundary value of the Φ^j-terms up to some suitable order $\geq j_0$ equal to the terms of corresponding order in the ϵ-expansions of (1.2).

The first result concerns existence of isolated, axially homogeneous Couette solutions for (1.1–2).

Theorem 1.1. *For $0 < \epsilon$, $0 < r_B - r_A$ small enough, there is an isolated axially homogeneous solution of (1.1–2). When ϵ tends to zero, the corresponding hydrodynamic moments of ϕ converge to solutions of the limiting fluid equations at the leading order ϵ.*

After a preliminary asymptotic analysis, this is proved in Section 2 and uses the techniques developed in.[1] It is followed in Sections 3–5 by a Taylor Couette bifurcation result.

Theorem 1.2. *For $0 < \epsilon$, $0 < r_B - r_A$ small enough, there is a smallest bifurcation value $u_{\theta Ab} > 0$, such that for $0 < u_{\theta A} \leq u_{\theta Ab}$ there exists an axially homogeneous solution to the problem (1.1–2), which at $u_{\theta Ab}$ bifurcates with a steady secondary solution appearing locally for $u_{\theta Ab} < u_{\theta A}$ which is axially symmetric and axially $(r_B - r_A)$−periodic. When ϵ tends to zero, the corresponding hydrodynamic moments converge to solutions of the limiting fluid equations at the leading order ϵ (which are of Taylor Couette type for the bifurcated solution).*

The positivity of such solutions is discussed in Section 5. In particular it is proved that

Theorem 1.3. *The solutions obtained in Theorems 1.1–2 are strictly positive in the case of Maxwellian molecules.*

The present section introduces the problem area and the main results together with the plan of the paper. The smallness condition on $r_B - r_A$ in the theorems can be removed by extending the asymptotic expansions to higher order.

Section 2 is devoted to the axially homogeneous case (cf ref.[1,13]) with a proof of Theorem 1.1. The presentation is brief, having much in common with similar results for the more involved context in ref.[1]. For the convenience of the reader, the bifurcation point for the Taylor rolls bifurcation is discussed in Section 3 through a fairly self-contained presentation. Section 4 considers the behaviour of the asymptotic expansion in a neighbourhood of the bifurcation point including a priori estimates. It uses a detailed Fourier expansion study, and builds on prior control of the fluid Taylor Couette situation – for an overview see ref.[4].

In Section 5 the rest term is studied. The hydrodynamic part considers each moment separately, and the proofs involve the detailed behaviour of the hydrodynamic terms in the asymptotic expansion. This is followed by an existence proof beyond the Taylor Couette bifurcation point via a contraction mapping based on the earlier a priori estimates. The final Section 6 studies the positivity of solutions to (1.1–2) using a related equation (see (6.1) below) with better positivity properties. The proof introduces a variant of (1.4), where for Maxwellian molecules the positivity of the asymptotic expansion $1+\varphi$ is under control. For numerical results related to this paper cf ref.[14].

Writing the solution of (1.1–2) as $f = M(1+\Phi)$, the new unknown $\Phi(r, z, v_r, v_\theta, v_z)$ should solve

$$v_r \frac{\partial \Phi}{\partial r} + v_z \frac{\partial \Phi}{\partial z} + \frac{1}{r} N\Phi = \frac{1}{\epsilon}(\tilde{L}\Phi + \tilde{J}(\Phi, \Phi)), \tag{1.5}$$

$$\Phi(1, v) = e^{\frac{1}{2}(v_\theta^2 - (v_\theta - \epsilon u_{\theta A1})^2)} - 1, \quad v_r > 0, \tag{1.6}$$

$$\Phi(r_B, v) = 0, \quad v_r < 0. \tag{1.7}$$

Here $\tilde{J}$ is the rescaled quadratic Boltzmann collision operator,

$$\tilde{J}(\Phi, \psi)(v) := \frac{1}{2} \int_{I\!R^3 \times S^2} B(v - v_*, \omega) M(v_*)(\Phi(v')\psi(v'_*) + \Phi(v'_*)\psi(v') - \Phi(v_*)\psi(v) - \Phi(v)\psi(v_*)) dv_* d\omega,$$

and $\tilde{L}$ is this operator linearized around 1,

$$(\tilde{L}\Phi)(v) := \int_{I\!R^3 \times S^2} B(v - v_*, \omega) M(v_*)(\Phi(v') + \Phi(v'_*) - \Phi(v_*) - \Phi(v)) dv_* d\omega = \tilde{K}(\Phi) - \tilde{\nu}\Phi.$$

Denoting by Φ_{Aj} the jth order coefficient of $\Phi(r_A)$ with respect to ϵ,

$$\Phi_{A1}(v) = u_{\theta A} v_\theta, \quad \Phi_{A2}(v) = \frac{u_{\theta A}^2}{2}\big(-1 + v_\theta^2\big), \tag{1.8}$$

$$\Phi_{A3}(v) = \frac{1}{2} u_{\theta A}^3 \left(-v_\theta + \frac{1}{3} v_\theta^3 \right). \tag{1.9}$$

2. THE AXIALLY HOMOGENEOUS SOLUTION

An axially homogeneous solution Φ will be determined as an approximate asymptotic expansion φ of order 2 with boundary values of first and second orders being Φ_{Ai}, $\Phi_{Bi}(= 0)$, $1 \le i \le 2$, plus a rest term ϵR,

$$\Phi(r, v) = \varphi(r, v) + \epsilon R(r, v),$$

with

$$\varphi(r, v) = \epsilon \Phi_{H1}(r, v) + \epsilon^2 \left(\Phi_{H2}(r, v) + \Phi_{K2A}\left(\frac{r-1}{\epsilon}, v \right) + \Phi_{K2B}\left(\frac{r - r_B}{\epsilon}, v \right) \right). \tag{2.1}$$

Recall (cf ref.[6]). that $\tilde{L}(v_\theta v_r \bar{B}) = v_\theta v_r$, $\tilde{L}(v_r \bar{A}) = v_r(v^2 - 5)$ for some functions $\bar{B}(|v|)$ and $\bar{A}(|v|)$, with $v_\theta v_r \bar{B}(|v|)$ and $v_r \bar{A}(|v|)$ bounded in the L^2_M-norm. Set $w_1 = \int v_r^2 v_\theta^2 \bar{B} M dv$, $w_2 = \int v_r^2 \bar{A} M dv$, $w_3 = \int v_r^2 v^2 \bar{A} M dv$.

In the asymptotic expansion the Hilbert terms Φ_{H1} and Φ_{H2} satisfy

$$\tilde{L}\Phi_{H1} = \tilde{L}\Phi_{H2} + \tilde{J}(\Phi_{H1}, \Phi_{H1}) - v \cdot \nabla_x \Phi_{H1} = 0.$$

They are given by

$$\Phi_{H1}(r, v) = b_1(r) v_\theta, \tag{2.2}$$

$$\Phi_{H2}(r, v) = a_2 + d_2 v^2 + b_2 v_\theta + c_2 v_r + \frac{1}{2} b_1^2 v_\theta^2 + \left(b_1' - \frac{1}{r} b_1 \right) v_r v_\theta \bar{B}, \tag{2.3}$$

where for compatibility reasons

$$b_1(r) = \frac{u_{\theta A}}{r_B^2 - 1}\left(\frac{r_B^2}{r} - r \right),$$

$$(a_2 + 5d_2)' + b_1 b_1' - \frac{1}{r} b_1^2 = 0, \quad c_2(r) = \frac{\gamma_2}{r}, \tag{2.4}$$

$$b_2'' + \frac{1}{r} b_2' - \frac{1}{r^2} b_2 = -\frac{1}{w_1}(b_1' + \frac{1}{r} b_1) c_2, \tag{2.5}$$

$$(w_3 - 5w_2)\left(d_2'' + \frac{1}{r}d_2'\right) = \left(b_1\left(b_1' - \frac{1}{r}b_1\right)\right)'$$

$$\int M u_n(u^2-5)(\tilde{L}^{-1}(2\tilde{J}(v_\theta, v_r v_\theta \bar{B}) - v_r(v_\theta^2 - 1))dv$$

$$+ \left(b_1 b_1' - \frac{1}{r}b_1^2\right)\int M(v^2-5)N(\tilde{L}^{-1}(2\tilde{J}(v_\theta, u_r v_\theta \bar{B}) - v_r(v_\theta^2-1))dv, \quad (2.6)$$

for some constant γ_2. The term l defined by

$$l = \frac{1}{\epsilon}\left(\tilde{L}\varphi + \tilde{J}(\varphi,\varphi) - \epsilon v \cdot \nabla_x \varphi\right), \quad (2.7)$$

is of ϵ-order two provided

$$\tilde{L}\Phi_{K2A} = v_r \frac{\partial \Phi_{K2A}}{\partial r}, \quad \tilde{L}\Phi_{K2B} = v_r \frac{\partial \Phi_{K2B}}{\partial r}.$$

With the term $(b_1' - \frac{1}{r}b_1)v_r v_\theta \bar{B}$, the function Φ_{H2} of (2.3–6) cannot satisfy the boundary conditions Φ_{A2} (resp. Φ_{B2}) at r_A (resp. r_B), and boundary layers are added. Denote by $\eta = \frac{r-r_A}{\epsilon}$ and $\mu = \frac{r-r_B}{\epsilon}$.

Proposition 2.1. *There are a second-order Hilbert term Φ_{H2} defined by (2.3) with a_2, d_2, b_2, c_2 satisfying (2.4-6), and Knudsen terms $\Phi_{K2A}(\eta, v)$, $\Phi_{K2B}(\mu, v)$ such that*

$$v_r \frac{\partial \Phi_{K2A}}{\partial \eta} = \tilde{L}\Phi_{K2A},$$
$$\Phi_{K2A}(0, v) = \Phi_{A2}(v) - \Phi_{H2}(r_A, v), \quad v_r > 0, \quad (2.8)$$
$$\lim_{\eta \to +\infty} \Phi_{K2A}(\eta, v) = 0,$$

and

$$v_r \frac{\partial \Phi_{K2B}}{\partial \mu} = \tilde{L}\Phi_{K2B},$$
$$\Phi_{K2B}(0, v) = \Phi_{B2}(v) - \Phi_{H2}(r_B, v), \quad v_r < 0, \quad (2.9)$$
$$\lim_{\mu \to -\infty} \Phi_{K2B}(\mu, v) = 0.$$

Proof of Proposition 2.1: By[2] and,[7] there are unique solutions ψ, ψ_{2A} and ψ_{2B} to

$$v_r \frac{\partial \psi}{\partial \eta} = \tilde{L}\psi,$$
$$\psi(0, v) = 0, \quad v_r > 0,$$

$$\int v_r \psi(\eta, v) M dv = 1,$$

$$v_r \frac{\partial \psi_{2A}}{\partial \eta} = \tilde{L}\psi_{2A},$$

$$\psi_{2A}(0, v) = -\left(b_1' - \frac{1}{r}b_1\right)(r_A) v_r v_\theta \bar{B} - \frac{1}{2}(u_{\theta A})^2 v_\theta^2, \quad v_r > 0,$$

$$\int v_r \psi_{2A}(\eta, v) M dv = 0,$$

$$v_r \frac{\partial \psi_{2B}}{\partial \eta} = \tilde{L}\psi_{2B},$$

$$\psi_{2B}(0, v) = -\left(b_1' - \frac{1}{r}b_1\right)(r_B) v_r v_\theta \bar{B}, \quad v_r > 0,$$

$$\int v_r \psi_{2B}(\eta, v) M dv = 0.$$

Moreover,

$$\lim_{\eta \to +\infty} \psi(\eta, v) = a_\infty + d_\infty v^2 + b_\infty v_\theta + v_r,$$

$$\lim_{\eta \to +\infty} \psi_{2A}(\eta, v) = a_{2A,\infty} + d_{2A,\infty} v^2 + b_{2A,\infty} v_\theta,$$

$$\lim_{\eta \to +\infty} \psi_{2B}(\eta, v) = a_{2B,\infty} + d_{2B,\infty} v^2 + b_{2B,\infty} v_\theta,$$

for some constants a_∞, d_∞, b_∞, $a_{2A,\infty}$, $d_{2A,\infty}$, $b_{2A,\infty}$, $a_{2B,\infty}$, $d_{2B,\infty}$ and $b_{2B,\infty}$. Choose

$$a_2(r_A) = \gamma_2 a_\infty + a_{2A,\infty} - \frac{1}{2}u_{\theta A}^2, \tag{2.10}$$

$$a_2(r_B) = -\frac{\gamma_2}{r_B} a_\infty + a_{2B,\infty}, \tag{2.11}$$

$$d_2(r_A) = \gamma_2 d_\infty + d_{2A,\infty}, \tag{2.12}$$

$$d_2(r_B) = -\frac{\gamma_2}{r_B} d_\infty + d_{2B,\infty}, \tag{2.13}$$

$$b_2(r_A) = \gamma_2 b_\infty + b_{2A,\infty}, \tag{2.14}$$

$$b_2(r_B) = \frac{\gamma_2}{r_B} b_\infty - b_{2B,\infty}, \tag{2.15}$$

$$e_2(r_A) = e_2(r_B) = 0. \tag{2.16}$$

Then

$$\Phi_{K2A} = \gamma_2(\psi - a_\infty - d_\infty v^2 - b_\infty v_\theta - v_r) + \psi_{2A} - a_{2A,\infty} - d_{2A,\infty} v^2 - b_{2A,\infty} v_\theta,$$

and

$$\Phi_{K2B}(\mu, v) = -\frac{\gamma_2}{r_B}(\psi(-\mu, -v) - a_\infty - d_\infty v^2 + b_\infty v_\theta + v_r) + \psi_{2B}(-\mu, -v) - a_{2B,\infty} - d_{2B,\infty} v^2 + b_{2B,\infty} v_\theta,$$

satisfy (2.8–9). The first equation. in (2.4) defines $a_2 + 5d_2$ if and only if

$$(a_2 + 5d_2)(r_B) - (a_2 + 5d_2)(r_A) = \frac{1}{2}u_{\theta A}^2 + \int_{r_A}^{r_B} \frac{1}{s} b_1^2(s)ds,$$

i.e.

$$\gamma_2 = \frac{r_B}{(r_B + 1)(a_\infty + 5d_\infty)}\left(a_{2B,\infty} - a_{2A,\infty} + 5d_{2B,\infty} - 5d_{2A,\infty} - \int_{r_A}^{r_B} \frac{1}{s} b_1^2(s)ds\right).$$

This fixes γ_2, hence c_2 and $a_2 + 5d_2$. Finally the second-order differential Eq. (2.5–6) together with the boundary conditions (2.12–15) define b_2 and d_2. □

Some properties of the axially homogeneous $f_\perp$ and $f_\parallel$ are now studied. As orthonormal basis for the kernel of $\tilde{L}$ in $L_M^2(I\!R^3)$ we take $\psi_0 = 1$, $\psi_\theta = v_\theta$, $\psi_r = v_r$, $\psi_z = v_z$, $\psi_4 = \frac{1}{\sqrt{6}}(v^2 - 3)$. For functions $f \in L_M^2([r_A, r_B] \times I\!R^3)$ we shall use the earlier splitting into $f = f_\parallel + f_\perp = P_0 f + (I - P_0) f$, such that

$$f_\parallel(r, v) = f_0(r) - \frac{\sqrt{6}}{2} f_4(r) + f_\theta(r) v_\theta + f_r(r) v_r + f_z(r) v_z + \frac{\sqrt{6}}{6} f_4(r) v^2,$$

$$\int M(v)(1, v, v^2) f_\perp(r, z, v)dv = 0,$$

$$\int M\psi_0 f(r, v)dv = f_0(r), \quad \int M\psi_4 f(r, v)dv = f_4(r),$$

$$\int M\psi_\theta f(r, v)dv = f_\theta(r), \quad \int M\psi_r f(r, v)dv = f_r(r),$$

$$\int M\psi_z f(r, v)dv = f_z(r).$$

Set $Df := v_r \frac{\partial f}{\partial r} + \frac{1}{r} N f$ with N given by (1.3). Due to the symmetries in the present setup, the position space may be changed from $\mathbb{R}^3$ with measure dx, to $\mathbb{R}^+$ with measure rdr. We may also take all functions even in v_z giving in particular $f_z = 0$. The relevant boundary space becomes

$$L^+ := \left\{ f; \mid f \mid_\sim = \left(\int_{v_r > 0} v_r M(v) \mid f(r_A, v) \mid^2 dv \right)^{\frac{1}{2}} + \left(\int_{v_r < 0} \mid v_r \mid M(v) \mid f(r_B, v) \mid^2 dv \right)^{\frac{1}{2}} < +\infty \right\}.$$

We shall use

$$\mathcal{W}^{q-}([r_A, r_B] \times \mathbb{R}^3) = \mathcal{W}^{q-} := \{ f; \tilde{\nu}^{\frac{1}{2}} f \in \tilde{L}^q, \tilde{\nu}^{-\frac{1}{2}} Df \in \tilde{L}^q, \gamma^+ f \in L^+ \}.$$

Define

$$f_{\theta^i r^j}(r) := \int M v_\theta^i v_r^j f_\perp(r, v) dv, \quad i + j \geq 2, \tag{2.17}$$

and $f_{\theta^i r^j 2}(r)$ correspondingly, when there is an extra factor $\mid v \mid^2$ in the integrand.

The following three propositions were already treated in the more involved context of ref.[1]. and the presentation here is accordingly brief.

Proposition 2.2. *Let $\tilde{\nu}^{-\frac{1}{2}} g \in \tilde{L}^q$, $F_b \in L^+$, $2 \leq q < \infty$, be given. There exists a unique solution $F \in \mathcal{W}^{q-}$ to*

$$DF = \frac{1}{\epsilon} (\tilde{L} F + 2 \sum_{j=1}^{j_1} \epsilon^j \tilde{J}(F, \Phi^j) + g), \quad F_{/\partial\Omega^+} = F_b, \tag{2.18}$$

where the terms Φ^j of the axially homogeneous asymptotic expansion were introduced above, and the boundary data F_b are given on the ingoing boundary $\partial\Omega^+$.

Define a specular reflection operator $\mathcal{S}$ at $r = r_A, \quad r_B$ as $\mathcal{S} f(r, v) = f(r, -v_r, v_\theta, v_z)$. We shall need the following estimates in $\tilde{L}^q$ for the non-hydrodynamic part $F_\perp$.

Proposition 2.3. *Let $2 \leq q \leq +\infty$, and let F be the solution in $\mathcal{W}^{q-}$ to (2.18) for $g = g_\perp$. The following estimates hold for small enough $\epsilon > 0$;*

$$\epsilon^{\frac{1}{2}} \mid \mathcal{S} F \mid_\sim + \mid \tilde{\nu}^{\frac{1}{2}} F_\perp \mid_2 \leq c(\mid \tilde{\nu}^{-\frac{1}{2}} g \mid_2 + \epsilon^{\frac{1}{2}} \mid F_b \mid_\sim + \epsilon(\parallel F_r \parallel_2 + \parallel F_\theta \parallel_2 + \parallel F_0 \parallel_2 + \parallel F_4 \parallel_2)), \tag{2.19}$$

$$\mid \tilde{\nu}^{\frac{1}{2}} F \mid_\infty \leq c(\mid \tilde{\nu}^{-\frac{1}{2}} g \mid_\infty + \epsilon^{-\frac{2}{q}} \mid \tilde{\nu}^{\frac{1}{2}} F \mid_q + \mid \tilde{\nu}^{\frac{1}{2}} F_b \mid_\sim). \tag{2.20}$$

The estimate (2.20) also holds when g has a non-vanishing hydrodynamic component $g_{\|}$.

Proof of Proposition 2.3: We first turn to the estimate (2.20). For $\varphi = 0$ and using [(12) p. 101], F can via a double iteration and splitting of the compact part K of $\tilde{L}$, be written as

$$F = U_\epsilon \frac{K'}{\epsilon} U_\epsilon \frac{K'}{\epsilon} F + Z_1 F + Z_2 g + Z_3 \gamma^+ F, \tag{2.21}$$

where

$$\left| \tilde{\nu}^{\frac{1}{2}} U_\epsilon \frac{K'}{\epsilon} U_\epsilon \frac{K'}{\epsilon} F \right|_\infty \leq c_\delta \epsilon^{-\frac{2}{q}} \mid \tilde{\nu}^{\frac{1}{2}} F \mid_q,$$

$$\left| \tilde{\nu}^{\frac{1}{2}} Z_1 F \right|_\infty \leq c\delta \left| \tilde{\nu}^{\frac{1}{2}} F \right|_\infty, \quad \left| \tilde{\nu}^{\frac{1}{2}} Z_2 g \right|_\infty \leq c \mid \tilde{\nu}^{-\frac{1}{2}} g \mid_\infty, \tag{2.22}$$

$$\left| \tilde{\nu}^{\frac{1}{2}} Z_3 \gamma^+ F \right|_\infty \leq c \mid \tilde{\nu}^{\frac{1}{2}} F_b \mid_\sim .$$

Using (2.21), (2.22) with δ small enough, gives (2.20). For ϵ small enough, the addition of $\tilde{J}(F, \varphi)$ to g does not change the result in this part of the proof, neither does the addition of a hydrodynamic component to g.

The mapping from $\tilde{\nu}^{-\frac{1}{2}} \tilde{L}^q \times L^+$ into $\mathcal{W}^{q-}$ given by $(g, F_b) \to F$, with F the solution to (2.18) for $\varphi = 0$, is continuous and bijective by [(12), Ch 6.1] for $2 \leq q \leq \infty$. Green's formula and the spectral inequality for $\tilde{L}$,

$$-\int M f \tilde{L} f dv \geq c \int M \tilde{\nu} f_\perp^2 dv$$

for some $c > 0$, give

$$\epsilon \mid \mathcal{S} F \mid_\sim^2 + \left| \tilde{\nu}^{\frac{1}{2}} F_\perp \right|_2^2 \leq \frac{c}{\delta} \left| \tilde{\nu}^{-\frac{1}{2}} g_\perp \right|_2^2 + \delta \left| \tilde{\nu}^{\frac{1}{2}} F_\perp \right|_2^2 + \epsilon \left| F_b \right|_\sim^2.$$

This completes the estimate (2.19) when $\varphi = 0$. The inclusion of $\tilde{J}(F, \varphi)$ to g, adds $c\epsilon \mid \tilde{\nu}^{\frac{1}{2}} F_\perp \mid_q$, which is incorporated in the left hand side, and a term

$$c\epsilon(\| F_r \|_2 + \| F_\theta \|_2 + \| F_0 \|_2 + \| F_4 \|_2). \qquad \square$$

Proposition 2.4. *Let* $g = g_{\|} + g_\perp$ *(i.e. with a possible hydrodynamic part* $g_{\|}$ *in g). Let F be the solution in* $\mathcal{W}^{2-}$ *to (2.18). For* $\epsilon > 0$ *and small enough,*

$$\| F_r \|_2 + \| F_\theta \|_2 + \| F_0 \|_2 + \| F_4 \|_2 \leq c(\mid F_\perp \mid_2 + \frac{1}{\epsilon} \mid \tilde{\nu}^{-\frac{1}{2}} g_\perp \mid_2 + \frac{1}{\epsilon^2} \mid g_{\|} \mid_2 + \mid F_b \mid_\sim). \tag{2.23}$$

Proof of Proposition 2.4: This follows from the proof in, ref.[1]. where ODE methods are used. A different Fourier based proof is given in Section 5 below. □

We next turn to the rest term. Given the asymptotic expansion φ of (2.1), the aim is to prove the existence of a rest term R, so that

$$f = M(1 + \varphi + \epsilon R) \tag{2.24}$$

is a solution to (1.1–2) with $M^{-1} f \in \tilde{L}^{\infty}$. This corresponds to the function R being a solution to

$$DR = \frac{1}{\epsilon}(\tilde{L}R + 2\tilde{J}(R, \varphi) + \epsilon \tilde{J}(R, R) + l),$$

with l defined in (2.7). Recall that the asymptotic expansion φ is of order two in ϵ with correct boundary values up to order two and that l of (2.7) – the pure φ-part of the equation - is of ϵ-order two in $\tilde{L}^q$. Notice that Φ^j may be constructed so that $D\Phi^j = (I - P_0)D\Phi^j$, hence that $l = l_\perp$.

Let the sequences $(R^n)_{n \in I\!N}$ be defined by $R^0 = 0$, and

$$DR^{n+1} = \frac{1}{\epsilon}\left(\tilde{L}R^{n+1} + 2\sum_{j=1}^{2} \epsilon^j \tilde{J}(R^{n+1}, \Phi^j) + g^n\right), \tag{2.25}$$

$$R^{n+1}(1, v) = R_A(v), \quad v_r > 0, \quad R^{n+1}(r_B, v) = R_B(v), \quad v_r < 0. \tag{2.26}$$

In (2.25–26)

$$g^n := \epsilon \tilde{J}(R^n, R^n) + l,$$

$$\epsilon R_A(v) := e^{\epsilon u_{\theta A1} v_\theta - \frac{\epsilon^2}{2} u_{\theta A1}^2} - 1 - \sum_{j=1}^{2} \epsilon^j \Phi^j(r_A, v), \quad v_r > 0,$$

$$\epsilon R_B(v) := -\sum_{j=1}^{2} \epsilon^j \Phi^j(r_B, v), \quad v_r < 0,$$

with R_A, R_B of ϵ-order two. We take $u_{\theta A} = U_{\theta A}(r_B - r_A)$, in order that $\eta = r_B - r_A$ be independent of an extra condition later imposed on $U_{\theta A}$ (beginning with (3.10) below). This makes the Φ^j of order $r_B - r_A$ throughout the paper.

For the rest term iteration scheme (2.25–26) the following holds.

Proposition 2.5. *For $0 < \epsilon$, $0 < r_B - r_A$ small enough, there is a unique sequence (R^n) of solutions to (2.25–26) in the set $X := \{R; | \tilde{\nu}^{\frac{1}{2}} R |_q \leq K\}$ for some constant K. The sequence converges in $\tilde{L}^q$ for $2 \leq q \leq \infty$, to an isolated solution of*

$$DR = \frac{1}{\epsilon}\left(\tilde{L}R + \epsilon \tilde{J}(R, R) + 2\tilde{J}(R, \varphi) + l\right), \tag{2.27}$$

$$R(1,v)=R_A(v),\quad v_r>0,\quad R(r_B,v)=R_B(v),\quad v_r<0. \tag{2.28}$$

When ϵ tends to zero, the corresponding hydrodynamic moments converge to solutions of the limiting fluid equations at the leading order ϵ.

Proof of Proposition 2.5: Denote by $\eta=r_B-r_A$. The existence result of Proposition 2.2 holds for the boundary value problem

$$Df=\frac{1}{\epsilon}\left(\tilde{L}f+2\tilde{J}(f,\varphi)+g\right),$$
$$f(1,v)=R_A(v),\quad v_r>0,\quad f(r_B,v)=R_B(v),\quad v_r<0.$$

Here $g=g_\perp$ and by Propositions 2.3–4

$$|\,\tilde{\nu}^{\frac12}f\,|_2\ \le\ c_1\left(\frac{1}{\epsilon}\,|\,\tilde{\nu}^{-\frac12}g_\perp\,|_2+|\,R_b\,|_\sim\right),$$
$$|\,\tilde{\nu}^{\frac12}f\,|_\infty\ \le\ c_1\left(|\,\tilde{\nu}^{-\frac12}g\,|_\infty+\frac{1}{\epsilon}\,|\,\tilde{\nu}^{\frac12}f\,|_2+|\,\tilde{\nu}^{\frac12}R_b\,|_\sim\right). \tag{2.29}$$

We recall that $|\,F_\parallel\,|_2\simeq|\,\tilde{\nu}^{\frac12}F_\parallel\,|_2$, and that for some constant c_2,

$$|\,\tilde{\nu}^{-\frac12}\tilde{J}(h,l)\,|_2\le c_2\,|\,\tilde{\nu}^{\frac12}h\,|_\infty|\,\tilde{\nu}^{\frac12}l\,|_2,\quad |\,\tilde{\nu}^{-\frac12}\tilde{J}(h,l)\,|_\infty\le c_2\,|\,\tilde{\nu}^{\frac12}h\,|_\infty|\,\tilde{\nu}^{\frac12}l\,|_\infty\,.$$

We will next show by induction that

$$|\,\tilde{\nu}^{\frac12}(R^{n+1}-R^n)\,|_2\le 16c_1^3c_2^2\eta\,|\,\tilde{\nu}^{\frac12}(R^n-R^{n-1})\,|_2,$$
$$|\,\tilde{\nu}^{\frac12}R^n\,|_\infty\le 8c_1^2c_2\eta,\ n\in I\!N. \tag{2.30}$$

For $n=0$, R^1 is the solution to

$$DR^1=\frac{1}{\epsilon}(\tilde{L}R^1+2\tilde{J}(\varphi,R^1)+l),$$
$$R^1(1,v)=R_A(v),\quad v_r>0,\quad R^1(r_B,v)=R_B(v),\quad v_r<0,$$

so that by (2.29) $|\,\tilde{\nu}^{\frac12}R^1\,|_2\le c_1c_2\eta\epsilon$, $|\,\tilde{\nu}^{\frac12}R^1\,|_\infty\le 2c_1^2c_2\eta$. Then, $R^{n+2}-R^{n+1}$ being solution to

$$D(R^{n+2}-R^{n+1})=\frac{1}{\epsilon}(\tilde{L}(R^{n+2}-R^{n+1})+2\tilde{J}(\varphi,R^{n+2}-R^{n+1})$$
$$+\epsilon\tilde{J}(R^{n+1}+R^n,R^{n+1}-R^n)),$$
$$R^{n+2}-R^{n+1}=0,\quad \partial\Omega^+,$$

satisfies

$$|\,\tilde{\nu}^{\frac12}(R^{n+2}-R^{n+1})\,|_2\ \le\ c_1\,|\,\tilde{\nu}^{-\frac12}J(R^{n+1}+R^n,R^{n+1}-R^n)\,|_2$$
$$\le c_1c_2(|\,\tilde{\nu}^{\frac12}R^{n+1}\,|_\infty+|\,\tilde{\nu}^{\frac12}R^n\,|_\infty)\,|\,\tilde{\nu}^{\frac12}(R^{n+1}-R^n)\,|_2$$
$$\le 16c_1^3c_2^2\eta\,|\,\tilde{\nu}^{\frac12}(R^{n+1}-R^n)\,|_2\,.$$

Moreover,

$$|\, \tilde{\nu}^{\frac{1}{2}} R^{n+2} \,|_\infty \leq |\, \tilde{\nu}^{\frac{1}{2}} (R^{n+2} - R^{n+1}) \,|_\infty + \ldots + |\, \tilde{\nu}^{\frac{1}{2}} (R^2 - R^1) \,|_\infty + |\, \tilde{\nu}^{\frac{1}{2}} R^1 \,|_\infty,$$

so that

$$|\, \tilde{\nu}^{\frac{1}{2}} R^{n+2} \,|_\infty \leq 8c_1^2 c_2 \eta,$$

for sufficiently small $\eta > 0$. And so (R^n) converges to some R, solution to (2.27–28) in $\tilde{L}^q$ for $q \leq \infty$. The contraction mapping construction guarantees that the solution is isolated. □

Proof of Theorem 1.1: The existence of isolated solutions to (1.1–2) is an immediate consequence of Proposition 2.5. It also follows that, for the corresponding solutions (1.4), when ϵ tends to zero the hydrodynamic moments converge to the (Hilbert type) corresponding leading (first) order limiting fluid solution given by (2.2). □

3. A PERIODIC BIFURCATION OF THE ASYMPTOTIC EXPANSION

Extend the asymptotic expansion of Section 2 by third and fourth order terms term $\Phi^3(r, v)$ and $\Phi^4(r, v)$, and denote it by

$$\epsilon b_1 v_\theta + \epsilon^2(\varphi_{2u} + \Phi_{K2A}(\eta, v) + \Phi_{K2B}(\mu, v)) + \epsilon^3(\varphi_{3u} + \Phi_{K3A}(\eta, v) + \Phi_{K3B}(\mu, v)) + \epsilon^4(\varphi_{4u} + \Phi_{K4A}(\eta, v) + \Phi_{K4B}(\mu, v)),$$

where $\varphi_{2u} = \Phi_{H2u}$ of Section 2. This expansion is uniform with respect to the variable z, and $\eta = \frac{r-1}{\epsilon}$, $\mu = \frac{r-r_B}{\epsilon}$. Consider the following z-periodic perturbation $\varphi(r, z, v)$ of the z-homogeneous expansion,

$$\begin{aligned}
\varphi(r, z, v) &= \epsilon \left(b_1 v_\theta + \delta \cos\alpha z (U v_\theta + V v_r) + \delta (\sin\alpha z) W v_z + \delta^2 U_{20} v_\theta \right) \\
&\quad + \epsilon^2 \left(\varphi_{2u} + \Phi_{K2A} + \Phi_{K2B} + \delta(\cos\alpha z)(\varphi_{11}^2 + \Phi_{K21A}(\eta, v) + \Phi_{K21B}(\mu, v)) \right. \\
&\quad + \delta(\sin\alpha z)(\psi_{11}^2 + \psi_{K21A} + \psi_{K21B}) + \delta^2(\varphi_{20}^2 + \Phi_{K20A} + \Phi_{K20B}) \\
&\quad + \delta^2(\cos 2\alpha z)(\varphi_{22}^2 + \Phi_{K22A} + \Phi_{K22B}) \\
&\quad \left. + \delta^2(\sin 2\alpha z)(\psi_{22}^2 + \psi_{K22A} + \psi_{K22B}) \right) \\
&\quad + \epsilon^3 \left(\varphi_{3u} + \Phi_{K3A} + \Phi_{K3B} + \delta(\cos\alpha z)(\varphi_{11}^3 + \Phi_{K31A} + \Phi_{K31B}) \right. \\
&\quad \left. + \delta(\sin\alpha z)(\psi_{11}^3 + \psi_{K31A} + \psi_{K31B}) \right) \\
&\quad + \epsilon^4(\phi_{4u} + \Phi_{K4A} + \Phi_{K4B}).
\end{aligned}$$

Here all coefficient functions are taken with respect to space as functions of r only. Look for boundary conditions where only the rotational velocity of first order in ϵ,

$b_1 + \delta(\cos\alpha z)U + \delta^2 U_{20}$, at $r_A = 1$ deviates from b_1 by a δ^2-order term $\Delta u_{\theta A}$. All the unknowns $U, V, W, \ldots$ should then vanish at r_A and r_B, except U_{20}, for which

$$U_{20}(r_A) = \Delta u_{\theta A}, \quad U_{20}(r_B) = 0.$$

Lemma 3.1. *Let*

$$l = \frac{1}{\epsilon}(L\varphi + J(\varphi, \varphi) - \epsilon v \cdot \nabla_x \varphi).$$

If $\delta \le \epsilon$ and if (U, V) are solutions to

$$L_\theta(U) + q_\theta V = 0, \quad L_r(V) + q_r U = 0,$$
$$U(r) = V(r) = V'(r) = 0 \quad \text{at} \quad r = r_A, r = r_B, \tag{3.1}$$

where

$$L_\theta(U) = U'' + \frac{1}{r}U' - \left(\frac{1}{r^2} + \alpha^2\right)U, \quad L_r(V) = V^{(4)} + \frac{2}{r}V^{(3)}$$
$$- \left(\frac{3}{r^2} + 2\alpha^2\right)V'' + \left(\frac{3}{r^3} - \frac{2\alpha^2}{r}\right)V' + (-\frac{3}{r^4} + \frac{2\alpha^2}{r^2} + \alpha^4)V,$$
$$q_\theta = \frac{2u_{\theta A}}{w_1(r_B^2 - 1)}, \quad q_r = -\frac{2\alpha^2 u_{\theta A}}{w_1(r_B^2 - 1)}\left(\frac{r_B^2}{r^2} - 1\right),$$

then the function φ can be taken z-dependent, and so that $l = l_\perp$ is of order ϵ^4 in $\tilde{L}^\infty$.

The function φ is the asymptotic expansion for an axially periodic solution bifurcating from the axially homogeneous one at $u_{\theta A} = u_{\theta Ab}$.

Proof of Lemma 3.1: Replacing in l, φ by its expansion implies that

$$l = \epsilon\delta\cos\alpha z\Bigg(L\big(\varphi_{11}^2 - b_1 U v_\theta^2 - b_1 V v_r v_\theta\big) - \big(U' - \frac{1}{r}U\big)v_r v_\theta$$
$$- \big(V' v_r^2 + \frac{1}{r}V v_\theta^2 + \alpha W v_z^2\big) + L\Phi_{K21A} - v_r\frac{\partial \Phi_{K21A}}{\partial \eta} + L\Phi_{K21B}$$
$$- v_r\frac{\partial \Phi_{K21B}}{\partial \mu}\Bigg) + \epsilon\delta\sin\alpha z\Bigg(L\big(\psi_{11}^2 - b_1 W v_\theta v_z\big) + \alpha U v_\theta v_z$$
$$+ (\alpha V - W')v_r v_z + L\psi_{K21A} - v_r\frac{\partial \psi_{K21A}}{\partial \eta} + L\psi_{K21B} - v_r\frac{\partial \psi_{K21B}}{\partial \mu}\Bigg)$$
$$+ \epsilon\delta^2\Bigg(L(\varphi_{20}^2 - \frac{1}{4}U^2 v_\theta^2 - \frac{1}{4}V^2 v_r^2 - \frac{1}{2}UV v_r v_\theta - \frac{1}{4}W^2 v_z^2 - b_1 U_{20} v_\theta^2)$$

$$- (U_{20}' - \frac{1}{r}U_{20})v_r v_\theta + L\Phi_{K20A} - v_r \frac{\partial \Phi_{K20A}}{\partial \eta} + L\Phi_{K20B} - v_r \frac{\partial \Phi_{K20B}}{\partial \mu}\Bigg)$$

$$+ \epsilon\delta^2 \cos 2\alpha z \Bigg(L(\varphi_{22}^2 - \frac{1}{4}U^2 v_\theta^2 - \frac{1}{4}V^2 v_r^2 - \frac{1}{2}UVv_r v_\theta + \frac{1}{4}W^2 v_z^2)$$

$$+ L\Phi_{K22A} - v_r \frac{\partial \Phi_{K22A}}{\partial \eta} + L\Phi_{K22B} - v_r \frac{\partial \Phi_{K22B}}{\partial \mu}\Bigg)$$

$$+ \epsilon\delta^2 \sin 2\alpha z \Bigg(L(\psi_{22}^2 - UWv_\theta v_z - VWv_r v_z) + L\psi_{K22A} - v_r \frac{\partial \psi_{K22A}}{\partial \eta}$$

$$+ L\psi_{K22B} - v_r \frac{\partial \psi_{K22B}}{\partial \mu}\Bigg) + \epsilon^2\delta \cos \alpha z \left(L\varphi_{11}^3 + 2J(b_1 v_\theta, \varphi_{11}^2) \right.$$

$$+ 2J(\varphi_{2u}, Uv_\theta + Vv_r) - (v_r \frac{\partial \varphi_{11}^2}{\partial r} + \frac{1}{r}N\varphi_{11}^2 + \alpha\psi_{11}^2 v_z) + L\Phi_{K31A}$$

$$+ 2J(b_1 v_\theta, \Phi_{K21A}) + 2J(Uv_\theta + Vv_r, \Phi_{K2A}) - N\Phi_{K21A} - v_r \frac{\partial \Phi_{K31A}}{\partial \eta}$$

$$+ L\Phi_{K31B} + 2J(b_1 v_\theta, \Phi_{K21B}) + 2J(Uv_\theta + Vv_r, \Phi_{K2B}) - \frac{1}{r_B}N\Phi_{K21B}$$

$$- v_r \frac{\partial \Phi_{K31B}}{\partial \mu}\Bigg) + \epsilon^2\delta \sin \alpha z \left(L\psi_{11}^3 + 2J(b_1 v_\theta, \psi_{11}^2) + 2J(\varphi_{2u}, Wv_z) \right.$$

$$- \left(v_r \frac{\partial \psi_{11}^2}{\partial r} + \frac{1}{r}N\psi_{11}^2 - \alpha\varphi_{11}^2 v_z \right) + L\psi_{K31A} + 2J(b_1 v_\theta, \psi_{K21A})$$

$$+ 2J(Wv_z, \Phi_{K2A}) - N\psi_{K21A} - v_r \frac{\partial \psi_{K31A}}{\partial \eta} + L\psi_{K31B} + 2J(b_1 v_\theta, \psi_{K21B})$$

$$+2J(Wv_z, \Phi_{K2B}) - \frac{1}{r_B}N\psi_{K21B} - v_r \frac{\partial \psi_{K31B}}{\partial \mu}\Bigg) + O(\epsilon^4).$$

The compatiblity conditions in the $\epsilon\delta \cos \alpha z$ term write

$$\alpha W = -V' - \frac{1}{r}V. \tag{3.2}$$

And so φ_{11}^2 can be taken as

$$\varphi_{11}^2 = a_{11}^2 + d_{11}^2 v^2 + b_{11}^2 v_\theta + c_{11}^2 v_r + e_{11}^2 v_z + b_1 U v_\theta^2 + b_1 V v_r v_\theta$$
$$+ (U' - \frac{1}{r}U)v_r v_\theta \bar{B} + \frac{1}{r}V(v_\theta^2 - v_r^2)\bar{B} + \alpha W(v_z^2 - v_r^2)\bar{B},$$

for some constants a^2_{11}, d^2_{11}, b^2_{11}, c^2_{11} and e^2_{11}. Moreover,

$$\psi^2_{11} = \alpha^2_{11} + \delta^2_{11} v^2 + \beta^2_{11} v_\theta + \gamma^2_{11} v_r + \eta^2_{11} v_z + b_1 W v_\theta v_z - \alpha U v_\theta v_z \bar{B} \\ - \alpha V v_r v_z \bar{B} + W' v_r v_z \bar{B},$$

for some constants α^2_{11}, δ^2_{11}, β^2_{11}, γ^2_{11} and η^2_{11}. Then, the compatibility conditions of the $\epsilon^2 \delta \cos \alpha z$-term of l are

$$(c^2_{11})' + \frac{1}{r}(c^2_{11}) + \alpha \eta^2_{11} = 0, \tag{3.3}$$

$$\frac{1}{w_1}(a^2_{11} + 5d^2_{11} + b_1 U)' = \alpha W' + \frac{2\alpha}{r} W + \frac{2}{r} V' + \left(\frac{2}{r^2} + \alpha^2\right) V + \frac{2}{w_1 r} b_1 U, \tag{3.4}$$

$$(b_1 V)' + \frac{2}{r} b_1 V + w_1 (U' - \frac{1}{r} U)' + \frac{2w_1}{r}(U' - \frac{1}{r} U) + \alpha b_1 W - \alpha^2 w_1 U = 0, \tag{3.5}$$

$$\alpha^2_{11} + 5\delta^2_{11} = 0. \tag{3.6}$$

Taking (3.2) into account in (3.5) implies that

$$L_\theta U + q_\theta V = 0.$$

The compatibility conditions of the $\epsilon^2 \delta \sin \alpha z$-term of l are

$$(\gamma^2_{11})' + \frac{1}{r}(\gamma^2_{11}) - \alpha e^2_{11} = 0, \tag{3.7}$$

$$(\alpha^2_{11} + 5\delta^2_{11})' = 0, \tag{3.8}$$

$$\frac{1}{w_1}(a^2_{11} + 5d^2_{11} + b_1 U) = W'' + \frac{1}{r} W' - 2\alpha^2 W - \alpha(V' + \frac{1}{r} V). \tag{3.9}$$

Differentiating (3.9) with respect to the variable r and taking (3.4) and (3.2) into account, implies that

$$L_r V + q_r U = 0.$$

It follows that the coefficients φ^2_{20}, φ^2_{22}, ψ^2_{22}, φ^3_{11}, ψ^3_{11}, as well as the Knudsen terms can be defined so that l be of order 4 provided (3.1) holds. □

Lemma 3.2. *Let $\alpha > 0$ be given. There are nonnegative functions u_1 and v_1, and $u_{\theta A} = u_{\theta Ab} > 0$, such that for $r_B - r_A$ small enough, the problem (3.1) has the solutions* $\{(U, V) = x(u_1, v_1); x \in I\!R\}$.

Proof of Lemma 3.2: The equation $L_\theta U = 0$ is disconjugate on $[1, r_B]$ for any $r_B > 1$ since

$$\int_1^{r_B} \left(r y'^2 + \left(\frac{1}{r} + \alpha^2 \right) y^2 \right) dr$$

is nonnegative ref.[5]. Hence there is a continuous Green function G such that for any continuous function f, the problem

$$L_\theta U = f, \quad U(1) = U(r_B) = 0,$$

has the unique solution

$$U(r) = \int_1^{r_B} G(r, s) f(s)\, ds.$$

Moreover,

$$G(r, s)(r - 1)(r - r_B) \geq 0, \quad (r, s) \in [1, r_B]^2,$$

so that G is non positive. It also satisfies

$$r G(r, s) = s G(s, r), \quad (r, s) \in [1, r_B]^2,$$

since

$$\int_1^{r_B} r L_\theta(U) X dr = \int_1^{r_B} r L_\theta(X) U dr.$$

By ref.[5] the equation

$$L_r(V) = 0, \quad V(1) = V(r_B) = V'(1) = V'(r_B) = 0,$$

is disconjugate on $[1, r_B]$ for $r_B - 1$ small enough. Hence there is a C^2 Green function H such that for any continuous function f, the problem

$$L_r V = f, \quad V(1) = V(r_B) = V'(1) = V'(r_B) = 0,$$

has the unique solution

$$V(r) = \int_1^{r_B} H(r, s) f(s)\, ds.$$

Moreover,

$$H(r, s)(r - 1)^2 (r - r_B)^2 \geq 0, \quad (r, s) \in [1, r_B]^2,$$

so that H is nonnegative. It also satisfies

$$r H(r, s) = s H(s, r), \quad (r, s) \in [1, r_B]^2,$$

since

$$\int_1^{r_B} r L_r(V) Y dr = \int_1^{r_B} r L_r(Y) V dr.$$

And so, solving (3.1) comes back to finding $u_{\theta Ab} := U_{\theta Ab}(r_B - 1)$ and $V_{11}^1 \geq 0$ such that

$$K V_{11}^1 = \left(\frac{w_1(r_B^2 - 1)}{4\alpha u_{\theta Ab}} \right)^2 V_{11}^1, \tag{3.10}$$

where K is the operator defined by

$$K V(r) = -\int_1^{r_B} \int_1^{r_B} H(r, s) \left(\frac{r_B^2}{s^2} - 1 \right) G(s, t) V(t)\, dt ds.$$

K is compact in $L^2(1, r_B)$. It maps the cone of the nonnegative functions of L^2 into its interior, since G is nonpositive, H is nonnegative, and neither G nor H are identically zero. And so the Krein-Rutman theorem applies. There is an eigenvector $v_1 \geq 0$ corresponding to a positive eigenvalue of K, $\left(\frac{w_1(r_B^2-1)}{4\alpha u_{\theta Ab}}\right)^2 = \left(\frac{w_1(r_B+1)}{4\alpha U_{\theta A}}\right)^2$ with algebraic and geometric multiplicity equal to one. Denote by

$$u_1(r) = -q_\theta \int_1^{r_B} G(r, s) v_1(s)\, ds, \quad r \in [1, r_B].$$

Then any $(x u_1, x v_1)$, $x \in I\!R_+$ is solution to (3.1). □

4. THE ASYMPTOTIC EXPANSION AROUND THE BIFURCATION POINT

If the trigonometric asymptotic expansion of Section 3 for z in a δ^2-neighbourhood of the bifurcation velocity $u_{\theta Ab}$ were used for the study of the rest term, then an extra restriction $\delta < \epsilon$ would be required, which in turn would prevent the study of hydrodynamic limits beyond the bifurcation point. Instead, a full δ-perturbation is now introduced. Fix the z-period to be $r_B - r_A$. Recall that the z-homogeneous Φ_{H1}, Φ_{H2}, and Φ_{H3} at the bifurcation point should satisfy

$$\tilde{L}\Phi_{H1} = \tilde{L}\Phi_{H2} + \tilde{J}(\Phi_{H1}, \Phi_{H1}) - v \cdot \nabla_x \Phi_{H1} = 0, \tag{4.1}$$

$$\tilde{L}\Phi_{H3} + 2\tilde{J}(\Phi_{H1}, \Phi_{H2}) - v \cdot \nabla_x \Phi_{H2} = 0, \tag{4.2}$$

$$\tilde{L}\Phi_{H4} + 2\tilde{J}(\Phi_{H1}, \Phi_{H3}) + \tilde{J}(\Phi_{H2}, \Phi_{H2}) - v \cdot \nabla_x \Phi_{H3} = 0. \tag{4.3}$$

The first condition in (4.1) implies

$$\Phi_{H1}(r, v) = a_1(r, z) + d_1(r, z)v^2 + b_1(r, z)v_\theta + c_1(r, z)v_r + e_1(r, z)v_z. \tag{4.4}$$

These first order terms satisfy the steady secondary Taylor Couette flow problem with correct boundary values, and are known to be smooth functions with uniform bounds in a δ^2-neighbourhood of $u_{\theta Ab}$. That problem was first rigorously studied in ref.[15]. using topological Leray Schauder degree, to be followed over the years by a number of alternative treatments and expansions – see ref.[(4)]. for references and an overview.

Denote by the index b when a z-homogeneous term Φ_{Hj} is evaluated at the bifurcation velocity $u_{\theta A} = u_{\theta Ab}$ of Lemma 3.1. With $\Phi_{Hj} = \Phi_{Hjb} + \delta\Phi_j^1$, $j = 1, 2, 3, 4$, and Φ_1^1 given by the smooth solution to the fluid Taylor Couette problem, we shall next construct $\Phi_2^1(x, v, \delta)$ so that (4.1–3) hold, i.e.

$$\tilde{L}\Phi_2^1 + g_{1\perp} - v_r\frac{\partial\Phi_1^1}{\partial r} - v_z\frac{\partial\Phi_1^1}{\partial z} - Nh_1 = 0, \tag{4.5}$$

$$\tilde{L}\Phi_3^1 + g_{2\perp} - v_r\frac{\partial\Phi_2^1}{\partial r} - v_z\frac{\partial\Phi_2^1}{\partial z} - Nh_2 = 0, \tag{4.6}$$

$$g_\perp - v_r\frac{\partial\Phi_3^1}{\partial r} - v_z\frac{\partial\Phi_3^1}{\partial z} - Nh_3 = 0, \tag{4.7}$$

with

$$g_{1\perp} = 2\tilde{J}(\Phi_{H1b}, \Phi_1^1) + \delta\tilde{J}(\Phi_1^1, \Phi_1^1), \quad h_1 = \frac{1}{r}\Phi_1^1,$$

$$g_{2\perp} = 2\tilde{J}(\Phi_{H1}, \Phi_2^1) + 2\tilde{J}(\Phi_{H2b}, \Phi_1^1), \quad h_2 = \frac{1}{r}\Phi_2^1,$$

$$g_\perp = \tilde{L}\Phi_4^1 + 2\tilde{J}(\Phi_{H1}, \Phi_3^1) + 2\tilde{J}(\Phi_{H3b}, \Phi_1^1) + 2\tilde{J}(\Phi_{H2b}, \Phi_2^1) + \delta\tilde{J}(\Phi_2^1, \Phi_2^1),$$

$$h_3 = \frac{1}{r}\Phi_3^1.$$

To provide the correct boundary values of the problem up to order three, we add boundary layer corrections to Φ_2^1 and Φ_3^1 of the type in Section 2, where we may use $\Phi_3^1 = \Phi_{3\perp}^1$. The previous boundary layer analysis of Section 2 based on ref.[(7)]. applies, when the equations are taken in Fourier space for the periodic z-variable. This is so since at the crucial steps in the decay study for the Milne problem in ref.[(7)]. the relevant squared L^2 – integrals in velocity space of the Fourier coefficients can be added to give (by Parseval's identity) analogous estimates for the corresponding squared L^2 – norms with respect to z of Φ_2^1 and Φ_3^1. This also holds for their z-derivatives, which in turn via Sobolev embedding leads to uniform bounds for the Knudsen layer terms with respect to z. Remaining Knudsen layer bounds, when in the uniform norm, follow as in Section 2.

The locally uniform smoothness of Φ_{H1} (for small δ), implies by (4.5) space-wise smoothness for $\Phi_{2,\perp}^1$ uniformly for small δ. The aim of this section is to prove by Fourier techniques, that the hydrodynamic moments of Φ_2^1 and its derivatives are uniformly bounded in L^∞ in a δ^2-neighbourhood of the bifurcation point $u_{\theta Ab}$ for small enough ϵ.

Introduce the linear change of variables from $(r, z) \in (1, r_B) \times (-\frac{r_B-1}{2}, \frac{r_B-1}{2})$ to $(s, Z) \in (-\pi, \pi)^2$, and Fourier expand functions F in the new variables (again denoted by (r, z)) as

$$F(r, z, v) = \sum_{(n,j)\in Z^2} \alpha^{nj}(v)e^{i(nr+jz)}.$$

In particular, the hydrodynamic moments F_0, F_4, F_r, F_θ, and F_z become

$$F_0(r, z) = \sum_{(n,j)} m_0^{nj} e^{i(nr+jz)}, \quad F_4(r, z) = \sum_{(n,j)} m_4^{nj} e^{i(nr+jz)},$$

$$F_r(r, z) = \sum_{(n,j)} u_r^{nj} e^{i(nr+jz)}, \quad F_\theta(r, z) = \sum_{(n,j)} u_\theta^{nj} e^{i(nr+jz)},$$

$$F_z(r, z) = \sum_{(n,j)} u_z^{nj} e^{i(nr+jz)},$$

where

$$m_0^{nj} := (\alpha^{nj}, 1), \quad m_4^{nj} := (\alpha^{nj}, \psi_4),$$

$$u_r^{nj} := (\alpha^{nj}, \psi_r), \quad u_\theta^{nj} := (\alpha^{nj}, \psi_\theta), \quad u_z^{nj} := (\alpha^{nj}, \psi_z).$$

Lemma 4.1. *For some $\delta_0 > 0$ and for $\eta = \frac{r_B - r_A}{2\pi}$ small enough, it holds that*

$$\| m_0(\Phi_2^1) \|_2 + \| m_4(\Phi_2^1) \|_2 + \| u_r(\Phi_2^1) \|_2 + \| u_\theta(\Phi_2^1) \|_2 + \| u_z(\Phi_2^1) \|_2 \leq c,$$

for $\delta < \delta_0$ and with c only depending on Φ_{H1} and the ingoing boundary values of Φ_2^1.

Proof of Lemma 4.1: Set $\lambda := (v_r^2 \bar{A}, \psi_4)$ and $w_1 := (v_r^2 v_\theta^2 \bar{B}, 1)$. Recall that $u_z^{n0} = 0$ due to the symmetry $F(z, v_z) = F(-z, -v_z)$. Notice that the Fourier coefficients of the first r-derivative contains a multiple of the boundary value difference,

$$\alpha^{nj}\left(\frac{\partial F}{\partial r}\right) = in\alpha^{nj}(F) + \frac{(-1)^n}{2\pi} d^j, \quad (n, j) \in Z^2,$$

whereas for the first z-derivative no such term is present. Set $d = (F(\pi - 0) - F(-\pi + 0))\frac{1}{2\pi}$ with d^j its j'th Fourier coefficient in the z-direction. Set

$$\begin{aligned}\Lambda_r^{nj} := {} & 3i(-1)^n d^j_{3,r^2} + i(v_r^2 - v_\theta^2, h_3^{nj}) + n(g_{2,\perp}^{nj}, (2v_r^2 - v_\theta^2 - v_z^2)\bar{B}) \\ & - in(-1)^n d^j_{2,v_r(2v_r^2 - v_\theta^2 - v_z^2)\bar{B}} + 3j(g_{2,\perp}^{nj}, v_r v_z \bar{B}) - 3ij(-1)^n d^j_{2,v_r^2 v_z \bar{B}} \\ & + 3ni((2v_r v_\theta^2 - v_r^3)\bar{B}, h_2^{nj}) - in^2(v_r(2v_r^2 - v_\theta^2 - v_z^2)\bar{B}, (I - P_0)\alpha_2^{nj})\end{aligned}$$

$$+3ji((v_\theta^2 v_z - v_r^2 v_z)\bar{B}, h_2^{nj}) - inj(v_z(2v_r^2 - v_\theta^2 - v_z^2)\bar{B}, (I-P_0)\alpha_2^{nj})$$
$$-3inj(v_r^2 v_z\bar{B}, (I-P_0)\alpha_2^{nj}) - 3ij^2(v_r v_z^2\bar{B}, (I-P_0)\alpha_2^{nj}), \tag{4.8}$$

$$\begin{aligned}\Lambda_z^{nj} := {} & 3i(-1)^n d_{3,rz}^j + i(v_r v_z, h_3^{nj}) + 3n(g_{2,\perp}^{nj}, v_r v_z\bar{B}) + ((-3in(-1)^n d^j_{2,v_r^2 v_z\bar{B}}\\ & - ij(-1)^n d^j_{2,v_r(2v_z^2-v_r^2-v_\theta^2)\bar{B}} + j(g_{2,\perp}^{nj}, (2v_z^2 - v_r^2 - v_\theta^2)\bar{B})\\ & + 3ni((v_\theta^2 v_z - v_r^2 v_z)\bar{B}, h_2^{nj}) - 3in^2(v_r^2 v_z\bar{B}, (I-P_0)\alpha_2^{nj})\\ & - 3ji(v_r v_z^2\bar{B}, h_2^{nj}) - 3inj(v_r v_z^2\bar{B}, (I-P_0)\alpha_2^{nj})\\ & - inj(v_r(2v_z^2 - v_r^2 - v_\theta^2)\bar{B}, (I-P_0)\alpha_2^{nj})\\ & - ij^2(v_z(2v_z^2 - v_r^2 - v_\theta^2)\bar{B}, (I-P_0)\alpha_2^{nj}).\end{aligned} \tag{4.9}$$

Let us first prove that for $(n, j) \neq (0, 0)$, and with Λ-indices one lower in (4.10), i.e. with (2,1) instead of (3,2),

$$\begin{aligned}m_{2,0}^{nj} = {} & \frac{4}{3}w_1(-1)^n d_{2,r}^j + (v_r, h_2^{nj}) + \sqrt{\frac{2}{3}}\frac{1}{\lambda(n^2+j^2)}\Big(-(-1)^n d^j_{3,v_r(v^2-5)}\\ & + in(g_{2,\perp}^{nj}, v_r\bar{A}) + ij(g_{2,\perp}^{nj}, v_z\bar{A}) + n^2((I-P_0)v_r^2\bar{A}, (I-P_0)\alpha_2^{nj})\\ & - (ij(-1)^n d^j_{2,v_r v_z\bar{A}} + in(-1)^n d^j_{2,v_r^2\bar{A}} + in(-1)^n((v_r^2 - v_\theta^2)\bar{A}, h_2^{nj})\\ & + ij(-1)^n(v_r v_z\bar{A}, h_2^{nj})) + j^2((I-P_0)v_z^2\bar{A}, (I-P_0)\alpha_2^{nj})\\ & + 2nj(v_r v_z\bar{A}, (I-P_0)\alpha_2^{nj}) - (v_r(v^2-5), h_3^{nj})\Big)\\ & + \frac{n}{3(n^2+j^2)}\Lambda_r^{nj} + \frac{j}{3(n^2+j^2)}\Lambda_z^{nj},\end{aligned} \tag{4.10}$$

$$\begin{aligned}m_{2,4}^{nj} = {} & \frac{1}{\lambda(n^2+j^2)}\Big((-1)^n d^j_{3,v_r(v^2-5)} - in(g_{2,\perp}^{nj}, v_r\bar{A}) - ij(g_{2,\perp}^{nj}, v_z\bar{A})\\ & + ij(-1)^n d^j_{2,v_r v_z\bar{A}} + in(-1)^n d^j_{2,v_r^2\bar{A}} + in(-1)^n((v_r^2 - v_\theta^2)\bar{A}, h_2^{nj})\\ & + ij(-1)^n(v_r v_z\bar{A}, h_2^{nj}) + (v_r(v^2-5), h_3^{nj})\\ & - n^2((I-P_0)v_r^2\bar{A}, (I-P_0)\alpha_2^{nj}) - j^2((I-P_0)v_z^2\bar{A}, (I-P_0)\alpha_2^{nj})\\ & - 2nj(v_r v_z\bar{A}, (I-P_0)\alpha_2^{nj})\Big),\end{aligned} \tag{4.11}$$

$$u^{nj}_{2,\theta} = \frac{1}{w_1(n^2+j^2)}\Big((-1)^n d^j_{3,r\theta} + 2(v_r v_\theta, h^{nj}_3) - in(g^{nj}_{2,\perp}, v_r v_\theta \bar{B})$$

$$- ij(g^{nj}_{2,\perp}, v_\theta v_z \bar{B}) - in(h^{nj}_2, (v^2_\theta - 2v^2_r)v_\theta \bar{B}) + 2ij(h^{nj}_2, v_\theta v_r v_z \bar{B})$$

$$+ in(-1)^n d^j_{2,v^2_r v_\theta \bar{B}} + ij(-1)^n d^j_{2,v_r v_\theta v_z \bar{B}} - n^2((I-P_0)v^2_r v_\theta \bar{B}, (I-P_0)\alpha^{nj}_2)$$

$$- j^2((I-P_0)v_\theta v^2_z \bar{B}, (I-P_0)\alpha^{nj}_2) - 2nj(v_r v_\theta v_z \bar{B}, (I-P_0)\alpha^{nj}_2)\Big), \tag{4.12}$$

$$u^{nj}_{2,r} = \frac{i}{3w_1(n^2+j^2)^2}(-j^2\Lambda^{nj}_r + nj\Lambda^{nj}_z) + \frac{in}{n^2+j^2}((-1)^n d^j_{2,r} + (v_r, h^{nj}_2)), \tag{4.13}$$

$$u^{nj}_{2,z} = \frac{i}{3w_1(n^2+j^2)^2}(nj\Lambda^{nj}_r - n^2\Lambda^{nj}_z) + \frac{ij}{n^2+j^2}((-1)^n d^j_{2,r} + (v_r, h^{nj}_2)). \tag{4.14}$$

□

Proof of (4.11): Take the scalar product of (4.7) with 1, $v^2 - 5$, v_θ, v_r and v_z, and identify the Fourier coefficients. For $(n, j) \in Z^2$ it leads to the following equations for the Φ_{H3}-coefficients

$$- i(-1)^n d^j_{3,r} + nu^{nj}_{3,r} + ju^{nj}_{3,z} - i(v_r, h^{nj}_3) = 0, \tag{4.15}$$

$$- i(-1)^n d^j_{3,v_r(v^2-5)} + n(v_r(v^2-5), \alpha^{nj}_3) + j(v_z(v^2-5), \alpha^{nj}_3)$$

$$- i(v_r(v^2-5), h^{nj}_3) = 0, \tag{4.16}$$

$$- i(-1)^n d^j_{3,r\theta} + n(v_r v_\theta, \alpha^{nj}_3) + j(v_\theta v_z, \alpha^{nj}_3) - i2(v_r v_\theta, h^{nj}_3) = 0, \tag{4.17}$$

$$- i(-1)^n d^j_{3,r^2} + n(v^2_r, \alpha^{nj}_3) + j(v_r v_z, \alpha^{nj}_3) - i(v^2_r - v^2_\theta, h^{nj}_3) = 0, \tag{4.18}$$

$$- i(-1)^n d^j_{3,rz} + n(v_r v_z, \alpha^{nj}_3) + j(v^2_z, \alpha^{nj}_3) - i(v_r v_z, h^{nj}_3) = 0. \tag{4.19}$$

Then take the scalar product of (4.6) with $v_r\bar{A}$ and $v_z\bar{A}$, and identify the Fourier coefficients,

$$(-1)^n d^j_{2,v^2_r\bar{A}} + in(v^2_r\bar{A}, \alpha^{nj}_2) + ij(v_r v_z\bar{A}, \alpha^{nj}_2) + ((v^2_r - v^2_\theta)\bar{A}, h^{nj}_2)$$

$$= (v_r(v^2-5), \alpha^{nj}_3) + (g^{nj}_{2,\perp}, v_r\bar{A}), \tag{4.20}$$

$$(-1)^n d^j_{2,v_r v_z \bar{A}} + in(v_r v_z \bar{A}, \alpha_2^{nj}) + ij(v_z^2 \bar{A}, \alpha_2^{nj}) + (v_r v_z \bar{A}, h_2^{nj})$$
$$= (v_z(v^2-5), \alpha_3^{nj}) + (g^{nj}_{2,\perp}, v_z \bar{A}). \tag{4.21}$$

Notice that

$$P_0(v_r^2 \bar{A}) = \lambda \psi_4, \quad P_0(v_r v_z \bar{A}) = 0, \quad P_0(v_z^2 \bar{A}) = \lambda \psi_4.$$

And so, (4.20), (4.21) write

$$(v_r(v^2-5), \alpha_3^{nj}) = -(g^{nj}_{2,\perp}, v_r \bar{A}) + i\lambda n m^{nj}_{2,4} + ((v_r^2 - v_\theta^2)\bar{A}, h_2^{nj}) + (-1)^n d^j_{2,v_r^2 \bar{A}}$$
$$+ in((I-P_0)v_r^2 \bar{A}, (I-P_0)\alpha_2^{nj}) + ij(v_r v_z \bar{A}, (I-P_0)\alpha_2^{nj}),$$
$$(v_z(v^2-5), \alpha_3^{nj}) = -(g^{nj}_{2,\perp}, v_z \bar{A}) + i\lambda j m^{nj}_{2,4} + (v_r v_z \bar{A}, h_2^{nj}) + (-1)^n d^j_{2,v_r v_z \bar{A}}$$
$$+ in(v_r v_z \bar{A}, (I-P_0)\alpha_2^{nj}) + ij((I-P_0)v_z^2 \bar{A}, (I-P_0)\alpha_2^{nj}).$$

Inserting these values of $(v_r(v^2-5), \alpha_3^{nj})$ and $(v_z(v^2-5), \alpha_3^{nj})$ in (4.16) provides (4.11). □

Proof of (4.12): It follows from the scalar product of (4.6) with $v_r v_\theta \bar{B}$ and $v_\theta v_z \bar{B}$, that

$$(-1)^n d^j_{2,v_r^2 v_\theta \bar{B}} + in(v_r^2 v_\theta \bar{B}, \alpha_2^{nj}) + ij(v_r v_\theta v_z \bar{B}, \alpha_2^{nj})$$
$$= (v_r v_\theta, \alpha_3^{nj}) + (g^{nj}_{2,\perp}, v_r v_\theta \bar{B}) + ((v_\theta^2 - 2v_r^2)v_\theta \bar{B}, h_2^{nj})$$
$$(-1)^n d^j_{2,v_r v_\theta v_z \bar{B}} + in(v_r v_\theta v_z \bar{B}, \alpha_2^{nj}) + ij(v_\theta v_z^2 \bar{B}, \alpha_2^{nj})$$
$$= (v_\theta v_z, \alpha_3^{nj}) + (g^{nj}_{2,\perp}, v_\theta v_z \bar{B}) - 2(v_\theta v_r v_z \bar{B}, h_2^{nj}).$$

Notice that

$$P_0(v_r^2 v_\theta \bar{B}) = w_1 v_\theta, \quad P_0(v_r v_\theta v_z \bar{B}) = 0, \quad P_0(v_\theta v_z^2 \bar{B}) = w_1 v_\theta.$$

And so,

$$(v_r v_\theta, \alpha_3^{nj}) = -(g^{nj}_{2,\perp}, v_r v_\theta \bar{B}) - ((v_\theta^2 - 2v_r^2)v_\theta \bar{B}, h_2^{nj}) + i w_1 n u^{nj}_{2,\theta}$$
$$+ (-1)^n d^j_{2,v_r^2 v_\theta \bar{B}} + in((I-P_0)v_r^2 v_\theta \bar{B}, (I-P_0)\alpha_2^{nj})$$
$$+ ij(v_r v_\theta v_z \bar{B}, (I-P_0)\alpha_2^{nj}),$$

$$(v_\theta v_z, \alpha_3^{nj}) = -(g^{nj}_{2,\perp}, v_\theta v_z \bar{B}) + 2(v_\theta v_r v_z \bar{B}, h_2^{nj}) + i w_1 j u^{nj}_{2,\theta} + (-1)^n d^j_{2,v_r v_\theta v_z \bar{B}}$$
$$+ in(v_r v_\theta v_z \bar{B}, (I-P_0)\alpha_2^{nj}) + ij((I-P_0)v_\theta v_z^2 \bar{B}, (I-P_0)\alpha_2^{nj}).$$

Inserting these values of $(v_r v_\theta, \alpha_3^{nj})$ and $(v_\theta v_z, \alpha_3^{nj})$ into (4.17) gives (4.12). □

Proof of (4.13–14): Equations (4.18–19) can also be written

$$-3i(-1)^n d^j_{3,r^2} - 3i(v_r^2 - v_\theta^2, h_3^{nj}) + n(v^2, \alpha_3^{nj}) + n(2v_r^2 - v_\theta^2 - v_z^2, \alpha_3^{nj})$$
$$+ 3j(v_r v_z, \alpha_3^{nj}) = 0, \tag{4.22}$$

$$-3i(-1)^n d^j_{3,rz} - 3i(v_r v_z, h_3^{nj}) + 3n(v_r v_z, \alpha_3^{nj}) + j(v^2, \alpha_3^{nj})$$
$$+ j(2v_z^2 - v_r^2 - v_\theta^2, \alpha_3^{nj}) = 0. \tag{4.23}$$

Also,

$$(v^2, \alpha_3^{nj}) = \sqrt{6} m^{nj}_{3,4} + 3m^{nj}_{3,0}.$$

It follows from the scalar product of (4.6) with $(2v_r^2 - v_\theta^2 - v_z^2)\bar{B}$ (resp.$(2v_z^2 - v_r^2 - v_\theta^2)\bar{B}$, $v_r v_z \bar{B}$) that

$$(2v_r^2 - v_\theta^2 - v_z^2, \alpha_3^{nj}) = -(g^{nj}_{2,\perp}, (2v_r^2 - v_\theta^2 - v_z^2)\bar{B}) + 4iw_1 n u^{nj}_{2,r}$$
$$+ (3v_r(v_r^2 - 2v_\theta^2)\bar{B}, h_2^{nj}) + (-1)^n d^j_{2, v_r(2v_r^2 - v_\theta^2 - v_z^2)\bar{B}}$$
$$+ in((I - P_0)v_r(2v_r^2 - v_\theta^2 - v_z^2)\bar{B}, (I - P_0)\alpha_2^{nj})$$
$$- 2iw_1 j u^{nj}_{2,z} + ij((I - P_0)v_z(2v_r^2 - v_\theta^2 - v_z^2)\bar{B}, (I - P_0)\alpha_2^{nj}),$$

$$(2v_z^2 - v_r^2 - v_\theta^2, \alpha_3^{nj}) = -(g^{nj}_{2,\perp}, (2v_z^2 - v_r^2 - v_\theta^2)\bar{B}) - 2iw_1 n u^{nj}_{2,r} + (3v_r v_z^2 \bar{B}, h_2^{nj})$$
$$+ (-1)^n d^j_{2, v_r(2v_z^2 - v_r^2 - v_\theta^2)\bar{B}} + in((I - P_0)v_r(2v_z^2 - v_r^2 - v_\theta^2)\bar{B}, (I - P_0)\alpha_2^{nj})$$
$$+ 4iw_1 j u^{nj}_{2,z} + ij((I - P_0)v_z(2v_z^2 - v_r^2 - v_\theta^2)\bar{B}, (I - P_0)\alpha_2^{nj}),$$

$$(v_r v_z, \alpha_3^{nj}) = -(g^{nj}_{2,\perp}, v_r v_z \bar{B}) + (v_z \bar{B}(v_r^2 - v_\theta^2), h_2^{nj}) + iw_1 n u^{nj}_{2,z} + iw_1 j u^{nj}_{2,r}$$
$$+ (-1)^n d^j_{2, v_r^2 v_z \bar{B}} + in((I - P_0)v_r^2 v_z \bar{B}, (I - P_0)\alpha_2^{nj})$$
$$+ ij((I - P_0)v_r v_z^2 \bar{B}, (I - P_0)\alpha_2^{nj}).$$

Inserting these values of $(2v_r^2 - v_\theta^2 - v_z^2, \alpha_3^{nj})$, $(2v_z^2 - v_r^2 - v_\theta^2, \alpha_3^{nj})$ and $(v_r v_z, \alpha_3^{nj})$ in (4.22–23) gives

$$Y_r^{nj} = \Lambda_r^{nj}, \quad Y_z^{nj} = \Lambda_z^{nj},$$

where

$$Y_r^{nj} := n(\sqrt{6}m_{3,4}^{nj} + 3m_{3,0}^{nj}) + iw_1(4n^2 + 3j^2)u_{2,r}^{nj} + iw_1 nju_{2,z}^{nj},$$

$$Y_z^{nj} := j(\sqrt{6}m_{3,4}^{nj} + 3m_{3,0}^{nj}) + iw_1 nju_{2,r}^{nj} + iw_1(3n^2 + 4j^2)u_{2,z}^{nj}.$$

This implies that

$$Y_r^{nj} - n\frac{nY_r^{nj} + jY_z^{nj}}{n^2 + j^2} = \Lambda_r^{nj} - n\frac{n\Lambda_r^{nj} + j\Lambda_z^{nj}}{n^2 + j^2},$$

which with

$$\frac{nY_r^{nj} + jY_z^{nj}}{n^2 + j^2} = \sqrt{6}m_{3,4}^{nj} + 3m_{3,0}^{nj} + 4iw_1(nu_{2,r}^{nj} + ju_{2,z}^{nj}), \tag{4.24}$$

gives

$$ju_{2,r}^{nj} - nu_{2,z}^{nj} = \frac{i}{3\gamma(n^2 + j^2)}(-j\Lambda_r^{nj} + n\Lambda_z^{nj}). \tag{4.25}$$

Taking the scalar product of (4.6) and 1, gives a second level version of (4.15). This together with (4.25) leads to (4.13–14). □

Proof of (4.10): It follows from (4.24) and (4.15) that

$$m_{3,0}^{nj} = -\sqrt{\frac{2}{3}}m_{3,4}^{nj} + \frac{4w_1}{3}((-1)^n d_{3,r}^j + (v_r, h_3^{nj})) + \frac{1}{3(n^2 + j^2)}(n\Lambda_r^{nj} + j\Lambda_z^{nj}).$$

This was based on (4.6), (4.7). A corresponding analysis based on (4.5), (4.6) gives

$$m_{2,0}^{nj} = -\sqrt{\frac{2}{3}}m_{2,4}^{nj} + \frac{4w_1}{3}((-1)^n d_{2,r}^j + (v_r, h_2^{nj})) + \frac{1}{3(n^2 + j^2)}(n\Lambda_r^{nj} + j\Lambda_z^{nj}),$$

(here with the indices in Λ_r and Λ_z correspondingly lowered from (3,2) to (2,1)) and (4.10) follows.

We now specialize g_j and h_j to those given by the problem, i.e. whose expressions follow (4.5–7). An a priori L^2-estimate of $\Phi_{2\perp}^1$ in terms of Φ_{H1} and independent of δ close to zero is immediate from the first equation. We shall next obtain a priori estimates for the hydrodynamic moments of Φ_2^1, that rely on Parseval's identity. Given (4.10–14) that will be used to control the $(n, j) \neq (0, 0)$ Fourier coefficients of the hydrodynamic moments of Φ_2^1, it remains to control the $(0, 0)$-Fourier coefficients.

First,

$$\alpha^{00}(\Phi_2^1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dz \left[\frac{1}{2}(\Phi_2^1(\pi-0)+\Phi_2^1(-\pi+0)) - \sum_{n\neq 0}\alpha^{n0}(\Phi_2^1)e^{in\pi}\right]$$

$$= \Delta_2 - \sum_{n\neq 0}\alpha^{n0}(\Phi_2^1)e^{in\pi}, \tag{4.26}$$

where $\Delta_2 = \frac{1}{4\pi}\int_{-\pi}^{\pi} dz[\Phi_2^1(\pi-0)+\Phi_2^1(-\pi+0)]$. Notice that $\Delta_{2,z}$ and $\Delta_{2,\theta}$ can be given in terms of integrals over $v_r > 0$ of $\Phi_{2,b}^1(-\pi)$ and of $\Phi_{2,\perp}^1$ and integrals over $v_r < 0$ of $\Phi_{2,b}^1(\pi)$ and of $\Phi_{2,\perp}^1$. They are obviously uniformly bounded for small $\delta > 0$. Three disjoint sets $A_j \subset \{v \in I\!R^3; v_r > 0\}$ can be so chosen that the three 3-vectors $(\int_{A_j}\psi_r M dv, \int_{A_j}\psi_0 M dv, \int_{A_j}\psi_4 M dv)$, $j = 1, 2, 3$, are linearly independent, and analogously for sets $B_i \subset \{v \in I\!R^3; v_r < 0\}$. Since the corresponding integrals of $\Phi_{2,b}^1(-\pi)$, $\Phi_{2,\perp}^1(-\pi)$, and $\Phi_{2,b}^1(\pi)$, $\Phi_{2,\perp}^1(\pi)$ are uniformly bounded for small $\delta > 0$, it follows that also $\Delta_{2,r}$, $\Delta_{2,0}$, $\Delta_{2,4}$ are uniformly bounded for small $\delta > 0$, and analogously for the hydrodynamic moments of d_2. Moreover, for any polynomial P, the moments

$$\int v_r P(v)\Phi_2^1(-\pi, z, v)M dz dv, \quad \text{and} \quad \int v_r P(v)\Phi_2^1(\pi, z, v)M dz dv,$$

are uniformly bounded w.r.t. δ, due to the Green formula. Bounds on such moments, as well as on Δ-moments and d-moments obtained in this way and uniform for small $\delta > 0$ will be denoted by C_0 below. □

Control of $m_4(\Phi_2^1)$. The coefficient $m_4^{00}(\Phi_2^1)$ is given by $c\left(\alpha_{v_r^2\bar{A}}^{00}(\Phi_2^1) - \alpha_{\perp,v_r^2\bar{A}}^{00}(\Phi_2^1)\right)$. Moreover,

$$\alpha_{v_r^2\bar{A}}^{00}(\Phi_2^1) = \Delta_{2,v_r^2\bar{A}} - \sum_{n\neq 0}\alpha_{v_r^2\bar{A}}^{00}(\Phi_2^1)(-1)^n,$$

where, by (4.16) and (4.20), for $n \neq 0$,

$$\alpha_{2,v_r^2\bar{A}}^{n0} = \frac{1}{in}((v_r(v^2-5), \alpha_3^{n0}) + (g_{2,\perp}^{n0}, v_r\bar{A}) - (-1)^n d_{2,v_r^2\bar{A}}^0$$
$$+((v_\theta^2 - v_r^2)\bar{A}, h_2^{n0}) = \frac{1}{in}\left(\frac{i}{n}(-1)^n d_{3,v_r(v^2-5)}^0 + \frac{i}{n}(v_r(v^2-5), h_3^{n0})\right.$$
$$\left.+(g_{2,\perp}^{n0}, v_r\bar{A}) - (-1)^n d_{2,v_r^2\bar{A}}^0 + ((v_\theta^2 - v_r^2)\bar{A}, h_2^{n0})\right).$$

Here it follows from (4.16) for $(n, j) = (0, 0)$ that $d_{3,v_r(v^2-5)}^0 = -(v_r(v^2-5), h_3^{00})$. This last term can be controlled like the other terms $\frac{1}{n^2}(v_r(v^2-5), h_3^{n0}) =$

$\frac{1}{n^2}(\frac{(v_r(v^2-5),\alpha_3)}{r})^{n0}$, $n \neq 0$, using their convolution form and the order of magnitude of the Fourier coefficients of $\frac{1}{r}$ together with (4.20). Hence,

$$|\alpha^{00}_{v_r^2\bar{A}}(\Phi_2^1)| \leq c(|g_{2,\perp}|_2 + C_0).$$

We now use (4.11) to sum the squares of the $m_4^{nj}(\Phi_2^1)$. There the upcoming sum

$$\sum_{(n,j)\neq(0,0)} \frac{(d^j_{3v_r(v^2-5)})^2}{(n^2+j^2)^2}$$

is estimated using again (4.16), (4.20–21) but without separating out the hydrodynamics. Namely, $d^j_{3v_r(v^2-5)}$ is computed from

$$\begin{aligned}\sum_n \frac{d^j_{3v_r(v^2-5)}}{(n^2+j^2)} = \sum_n \frac{(-1)^n}{n^2+j^2}\{&in(g^{nj}_{2,\perp}, v_r\bar{A}) - in(-1)^n d^j_{2,v_r^2\bar{A}} + n^2(v_r^2\bar{A}, \alpha_2^{nj}) \\ &+ 2inj(v_r v_z\bar{A}, \alpha_2^{nj}) + j(g^{nj}_{2,\perp}, v_z\bar{A}) - ij(-1)^n d_{2,v_r v_z\bar{A}} \\ &+ j^2(v_z^2\bar{A}, \alpha_2^{nj}) - (v_r(v^2-5), h_3^{nj}) - in((v_r^2 - v_\theta^2)\bar{A}, h_2^{nj}) \\ &- ij(v_r v_z\bar{A}, h_2^{nj})\},\end{aligned}$$

and here $\sum_n(v_r^2\bar{A}, \alpha_2^{nj})(-1)^n$ equals the boundary term $\Delta^j_{2,v_r^2\bar{A}}$. The same idea is used to estimate upcoming d_3-terms in the other hydrodynamic α_2-moments below, i.e. the idea to replace such d_3-moment by α_2 boundary moments plus easily tractable terms. Notice that $(j^2(v_r^2 - v_z^2)\bar{A}, \alpha_2^{nj})$ is a non-hydrodynamic moment, and that the factor in front of $d^j_{v_r(v^2-5)}$ after summation has magnitude j^{-1}. For the term $\frac{(v_r(v^2-5),h_3^{nj})}{n^2+j^2}$ we again use its convolution form together with (4.20). The scaling from $r_B - r_A$ to 2π at the beginning of the section, introduces an extra factor η into the equations, which is explicitly accounted for in the related Section 5 below. In the present section η can be considered as a factor in $h, g, \tilde{L}$. Using it and adding up, for small enough $\eta > 0$ it follows that

$$\begin{aligned}\| m_4(\Phi_2^1) \|_2^2 &\leq c\left(| g_{2,\perp} |_2^2 + | \Phi^1_{2\perp} |_2^2 + |\Delta_{2,v_r^2\bar{A}}|^2 + \sum_j d^{j2}_{2,v_r v_z\bar{A}} + \sum_j d^{j2}_{2,v_r^2\bar{A}} \right) \\ &= c\left(| g_{2,\perp} |_2^2 + | \Phi^1_{2\perp} |_2^2 + C_0\right), \qquad (4.27)\end{aligned}$$

uniformly for δ in a right neighbourhood of zero.

Control of $m_0(\Phi_2^1)$

First,

$$(\alpha^{00}(\Phi_2^1), v_r^2) = \frac{\sqrt{6}}{3} m_4^{00}(\Phi_2^1) + m_0^{00}(\Phi_2^1) + ((I - P_0)v_r^2, \alpha^{00}(\Phi_2^1)),$$

and we use the former control of $m_4^{00}(\Phi_2^1)$, (4.26) for $\alpha^{00}(\Phi_2^1)$, and the 2-version of (4.18) for (α_2^{n0}, v_r^2), i.e.

$$(\alpha^{n0}(\Phi_2^1), v_r^2) = \frac{i}{n}((-1)^n d_{2,r^2}^0 + (v_r^2 - v_\theta^2, h_2^{n0})).$$

Then, for $(n, j) \neq (0, 0)$, equation similar to (4.18–19), but obtained from Eq. (4.6), imply that, for Φ_2^1,

$$n(m_0^{nj}(\Phi_2^1) + \sqrt{\frac{2}{3}} m_4^{nj}(\Phi_2^1)) + n(v_r^2, (I - P_0)\alpha^{nj}(\Phi_2^1)) + j(v_r v_z, (I - P_0)\alpha^{nj}(\Phi_2^1))$$
$$= i((v_\theta^2 - v_r^2, h_2^{nj}) + (-1)^n d_{2,r^2}^j),$$

$$j(m_0^{nj}(\Phi_2^1) + \sqrt{\frac{2}{3}} m_4^{nj}(\Phi_2^1)) + j(v_z^2, (I - P_0)\alpha^{nj}(\Phi_2^1)) + n(v_r v_z, (I - P_0)\alpha^{nj}(\Phi_2^1))$$
$$= i((v_r v_z, h_2^{nj}) + (-1)^n d_{2,rz}^j).$$

Multiplying the former by $\frac{n}{n^2+j^2}$ and the latter by $\frac{j}{n^2+j^2}$, implies that

$$m_0^{nj}(\Phi_2^1) = -\left(\sqrt{\frac{2}{3}} m_4^{nj}(\Phi_2^1) + \frac{n^2}{n^2 + j^2}(v_r^2, (I - P_0)\alpha^{nj}(\Phi_2^1))\right.$$
$$\left. + \frac{j^2}{n^2 + j^2}(v_z^2, (I - P_0)\alpha^{nj}(\Phi_2^1)) + 2\frac{nj}{n^2 + j^2}(v_r v_z, (I - P_0)\alpha^{nj}(\Phi_2^1))\right)$$
$$+ \frac{n}{n^2 + j^2} i\left((-1)^n d_{2,r^2}^j + (v_\theta^2 - v_r^2, h_2^{nj})\right)$$
$$+ \frac{ij}{n^2 + j^2}\left((v_r v_\theta, h_2^{nj}) + (-1)^n d_{2,rz}^j\right).$$

We thus obtain

$$\| m_0(\Phi_2^1) \|_2^2 \leq c\Bigg(| g_{2,\perp} |_2^2 + | \Phi_{2,\perp}^1 |_2^2 + |\Delta_{2,v_r^2 \bar{A}^2}| + |\Delta_{2,r^2}^2|$$
$$+ \sum_j d_{2,v_r v_z \bar{A}}^{j2} + \sum_j d_{2,v_r^2 \bar{A}}^{j2} + \sum_j d_{2,r^2}^{j2} + \sum_j d_{2,rz}^{j2}\Bigg)$$
$$= c\left(| g_{2,\perp} |_2^2 + | \Phi_{2,\perp}^1 |_2^2 + C_0\right). \tag{4.28}$$

Control of $u_\theta(\Phi_2^1)$

Similarly to the m_4-case,

$$\| u_\theta(\Phi_2^1) \|_2^2 \leq c\Bigg(| g_{2\perp} |_2^2 + | \Phi_{2,\perp}^1 |_2^2 + |\Delta_{2,v_r^2 v_\theta \bar{B}}|^2 + \sum_j d_{2,v_r^2 v_\theta \bar{B}}^{j2}$$

$$+ \sum_j d_{2,v_r v_\theta v_z \bar{B}}^{j2}\Bigg) = c\left(| g_{2\perp} |_2^2 + | \Phi_{2,\perp}^1 |_2^2 + C_0\right). \quad (4.29)$$

Control of $u_r(\Phi_2^1)$ and $u_z(\Phi_2^1)$.

First,

$$u_r^{00}(\Phi_2^1) = \Delta_{2,r} - \sum_{n\neq 0} u_r^{n0}(\Phi_2^1),$$

$$u_z^{00}(\Phi_2^1) = \Delta_{2,z} - \sum_{n\neq 0} u_z^{n0}(\Phi_2^1),$$

$$| \Delta_{2,r} | \leq C_0, \quad | \Delta_{2,z} | \leq C_0.$$

Then, for $h_2 = 0$, a direct estimate using (4.13-14) and (4.26) gives

$$\| u_r(\Phi_2^1) \|_2^2 + \| u_z(\Phi_2^1) \|_2^2 \leq c\Bigg(| g_{2\perp} |_2^2 + | \Phi_{2,\perp}^1 |_2^2$$

$$+ \sum_{(j,n)\neq(0,0)} \frac{1}{(n^2+j^2)^3}(j d_{3,r^2}^j - n d_{3,rz}^j)^2 + C_0\Bigg).$$

Also here the d_3-boundary terms can be removed. Namely, $j d_{3,r^2}^j - n d_{3,rz}^j$ can be estimated using (4.22–23), and the appearing $\alpha_{3,\perp}^{nj}$ by (4.6). This gives

$$\| u_r(\Phi_2^1) \|_2^2 + \| u_z(\Phi_2^1) \|_2^2 \leq c\left(| g_{2\perp} |_2^2 + | \Phi_{2,\perp}^1 |_2^2 + C_0\right). \quad (4.30)$$

Changing to $h_2 = \frac{\Phi_2^1}{r}$ and after some additional computations of the previous type, the a priori estimates for the v_r- and v_z-terms also follow in the $\frac{N}{r}\Phi_2^1$-setting.

With g_j and h_j, $j = 1, 2, 3$, defined immediately after (4.7), it follows that Φ_2^1 is bounded in L^2 uniformly for small $\delta > 0$. Namely, the hydrodynamic estimates follow from (4.27–30), whereas (4.5) after multiplication with Φ_2^1 and integration implies that

$$| \tilde{\nu}^{\frac{1}{2}} \Phi_{2\perp}^1 |_2 \leq C(\Phi_{H1}),$$

where by the uniform in δ bounds on Φ_{H1}, the constant in the right hand side is independent of δ. As an end result

$$| \tilde{\nu}^{\frac{1}{2}} \Phi_2^1 |_2 \leq C(\Phi_{H1}, \Phi_b^2), \tag{4.31}$$

uniformly in δ. A similar analysis with similar results can be carried out for the space derivatives of Φ_2^1, which satisfy equations of the type (4.5–7) including some additional already known terms. By the Sobolev embedding theorem it follows that the moments of Φ_2^1 in velocity space are bounded and continuous together with their space derivatives. (Obviously the above analysis can also be used for a contracting iteration construction of Φ_2^1.)

5. THE REST TERM

With the same change of variables as in Section 4, namely from $(r, z) \in (1, r_B) \times (-\frac{r_B-1}{2}, \frac{r_B-1}{2})$ to $(s, Z) \in (-\pi, \pi)^2$, we will for the iteration procedure for the rest term be interested in the case when the new unknown $\tilde{F}(s, Z, v) := F(\eta s + \frac{r_B+1}{2}, \eta Z, v)$ solves

$$v_r \frac{\partial \tilde{F}}{\partial s} + v_z \frac{\partial \tilde{F}}{\partial Z} + \eta \nu(s) N \tilde{F} = \frac{\eta}{\epsilon}(\tilde{L}\tilde{F} + \tilde{g}), \tag{5.1}$$

where $\nu(s) = \frac{2}{2\eta s + r_B + 1}$. The control of the hydrodynamic moments will again be obtained by Fourier series expansions. Write (in the new variables) the Fourier expanded density function $\tilde{F}$ as

$$\tilde{F}(s, Z, v) = \sum_{(n,j)\in Z^2} \alpha^{nj}(v) e^{i(ns+jZ)}.$$

The hydrodynamic moments $\tilde{F}_0$, $\tilde{F}_4$, $\tilde{F}_r$, $\tilde{F}_\theta$, and $\tilde{F}_z$ become

$$\tilde{F}_0(s, Z) = \sum_{(n,j)} m_0^{nj} e^{i(ns+jZ)}, \quad \tilde{F}_4(s, Z) = \sum_{(n,j)} m_4^{nj} e^{i(ns+jZ)},$$

$$\tilde{F}_r(s, Z) = \sum_{(n,j)} u_r^{nj} e^{i(ns+jZ)}, \quad \tilde{F}_\theta(s, Z) = \sum_{(n,j)} u_\theta^{nj} e^{i(ns+jZ)},$$

$$\tilde{F}_z(s, Z) = \sum_{(n,j)} u_z^{nj} e^{i(ns+jZ)},$$

where

$$m_0^{nj} := (\alpha^{nj}, 1), \quad m_4^{nj} := (\alpha^{nj}, \psi_4),$$

$$u_r^{nj} := (\alpha^{nj}, \psi_r), \quad u_\theta^{nj} := (\alpha^{nj}, \psi_\theta), \quad u_z^{nj} := (\alpha^{nj}, \psi_z).$$

Recall that (α, β) denotes the scalar product $\int \alpha(v)\beta(v)M(v)dv$, and notice that $u_z^{n0} = 0$ due to the symmetry $\tilde{F}(s, Z, v_r, v_\theta, v_z) = \tilde{F}(s, -Z, v_r, v_\theta, -v_z)$. Set $d = (\tilde{F}(\pi - 0) - \tilde{F}(-\pi + 0))\frac{1}{2\pi}$ with d^j its j'th Fourier coefficient in the Z-direction.

Denote by $\lambda := (v_r^2 \bar{A}, \psi_4)$ and by $Q = I - P_0$, and write

$$e(Z, v) := \frac{1}{2\pi}\left((v\tilde{F})(\pi - 0, Z, v) - (v\tilde{F})(-\pi + 0, Z, v)\right) = \sum_{j \in Z} e^j(v) e^{ijZ}.$$

Set

$$\begin{aligned}
\Lambda_r^{nj} :=& -\frac{3i}{\epsilon}(g^{nj}, v_r) - 3j(g^{nj}, v_r v_z \bar{B}) + n(g^{nj}, (2v_r^2 - v_\theta^2 - v_z^2)\bar{B}) \\
&- in\epsilon(-1)^n d^j_{v_r(2v_r^2 - v_\theta^2 - v_z^2)\bar{B}} + 3i(-1)^n d^j_{r^2} - 3ij\epsilon(-1)^n d^j_{v_r^2 v_z \bar{B}} \\
&- i\epsilon n^2(Qv_r(2v_r^2 - v_\theta^2 - v_z^2)\bar{B}, Q\alpha^{nj}) - i\epsilon nj(Qv_z(2v_r^2 - v_\theta^2 - v_z^2)\bar{B}, Q\alpha^{nj}) \\
&- 3i\epsilon nj(Qv_r^2 v_z \bar{B}, Q\alpha^{nj}) - i\epsilon j^2(Qv_r v_z^2 \bar{B}, Q\alpha^{nj}) \\
&- \epsilon\eta n(v\tilde{F})^{nj}_{v_r(2v_r^2 - 7v_\theta^2 - v_z^2)\bar{B}} - 3\epsilon\eta j(v\tilde{F})^{nj}_{(v_r^2 - v_\theta^2)\bar{B}} + 3i\eta(v\tilde{F})^{nj}_{v_r^2 - v_\theta^2},
\end{aligned}$$

$$\begin{aligned}
\Lambda_z^{nj} :=& -\frac{3i}{\epsilon}(g^{nj}, v_z) + j(g^{nj}, (2v_z^2 - v_r^2 - v_\theta^2)\bar{B}) + 3n(g^{nj}, v_r v_z \bar{B}) \\
&- 3in\epsilon(-1)^n d^j_{v_r^2 v_z \bar{B}} - ij\epsilon(-1)^n d^j_{v_r(2v_z^2 - v_r^2 - v_\theta^2)\bar{B}} + 3i(-1)^n d^j_{rz} \\
&- 3i\epsilon n^2(Qv_r^2 v_z \bar{B}, Q\alpha^{nj}) - 3i\epsilon nj(Qv_r v_z^2 \bar{B}, Q\alpha^{nj}) \\
&- i\epsilon nj(Qv_r(2v_z^2 - v_r^2 - v_\theta^2)\bar{B}, Q\alpha^{nj}) - i\epsilon j^2(Qv_z(2v_z^2 - v_r^2 - v_\theta^2)\bar{B}, Q\alpha^{nj}) \\
&- 3\epsilon\eta n(v\tilde{F})^{nj}_{(v_r^2 - v_\theta^2)\bar{B}} - \epsilon\eta j(v\tilde{F})^{nj}_{v_r(-v_r^2 - v_\theta^2 + 2v_z^2)\bar{B}} + i\eta(v\tilde{F})^{nj}_{v_r v_z}.
\end{aligned}$$

Lemma 5.1. *Let $\tilde{F}$ be a solution to (5.1). Denote by $\epsilon_1 = \frac{\epsilon}{\eta}$. For $(n, j) \neq (0, 0)$,*

$$\begin{aligned}
m_0^{nj} =& -\frac{4}{3}w_1(g^{nj}, 1) + \frac{4}{3}w_1(-1)^n d_r^j + \frac{n\Lambda_r + j\Lambda_z}{3(n^2 + j^2)} + \frac{4}{3}w_1(v\tilde{F})^{nj}_{v_r} \\
&+ \sqrt{\frac{2}{3}}\frac{1}{\lambda(n^2 + j^2)}\left(\frac{1}{\epsilon_1^2}(g^{nj}, v^2 - 5) + i\frac{n}{\epsilon_1}g^{nj}_{v_r \bar{A}} + \frac{ij}{\epsilon_1}(g^{nj}, v_z \bar{A}) - \frac{\eta}{\epsilon_1}(vg)^{nj}_{v_r \bar{A}}\right. \\
&- \frac{(-1)^n}{\epsilon_1}d^j_{v_r(v^2 - 5)} - i(-1)^n n d^j_{v_r^2 \bar{A}} - i(-1)^n j d^j_{v_r v_z \bar{A}} + \eta(-1)^n e^j_{v_r^2 \bar{A}} \\
&+ n^2(Qv_r^2 \bar{A}, Q\alpha^{nj}) + j^2(Qv_z^2 \bar{A}, Q\alpha^{nj}) + 2nj(v_r v_z \bar{A}, Q\alpha^{nj})
\end{aligned}$$

$$-i\eta n(\nu\tilde{F})^{nj}_{v_\theta^2-v_r^2} - i\eta j(\nu\tilde{F})^{nj}_{v_r v_z} + i\eta j(\nu\tilde{F})^{nj}_{v_r v_z\bar{A}}$$

$$-\eta^2(\nu^2\tilde{F})^{nj}_{v_r^2-v_\theta^2} + i\eta n(\nu\tilde{F})^{nj}_{v_r^2\bar{A}} - \eta(\nu'\tilde{F})^{nj}_{v_r^2\bar{A}}\Big), \tag{5.2}$$

$$m_4^{nj} = \frac{1}{\lambda(n^2+j^2)}\Big(-\frac{1}{\epsilon_1^2}(g^{nj}, v^2-5) - i\frac{n}{\epsilon_1}g^{nj}_{v_r\bar{A}} - \frac{ij}{\epsilon_1}(g^{nj}, v_z\bar{A}) + \frac{\eta}{\epsilon_1}(\nu g)^{nj}_{v_r\bar{A}}$$

$$+\frac{(-1)^n}{\epsilon_1}d^j_{v_r(v^2-5)} + i(-1)^n n d^j_{v_r^2\bar{A}} + ij(-1)^n d^j_{v_r v_z\bar{A}} - \eta(-1)^n e^j_{v_r^2\bar{A}}$$

$$-n^2(Qv_r^2\bar{A}, Q\alpha^{nj}) - j^2(Qv_z^2\bar{A}, Q\alpha^{nj}) - 2nj(v_r v_z\bar{A}, Q\alpha^{nj})$$

$$+i\eta n(\nu\tilde{F})^{nj}_{v_\theta^2-v_r^2} + i\eta j(\nu\tilde{F})^{nj}_{v_r v_z} - i\eta j(\nu\tilde{F})^{nj}_{v_r v_z\bar{A}} + \eta^2(\nu\tilde{F})^{nj}_{v_r^2-v_\theta^2}$$

$$-i\eta n(\nu\tilde{F})^{nj}_{v_r^2\bar{A}} + \eta(\nu'\tilde{F})^{nj}_{v_r^2\bar{A}}\Big), \tag{5.3}$$

$$u_\theta^{nj} = \frac{1}{w_1(n^2+j^2)}\Big(-\frac{1}{\epsilon_1^2}(g^{nj}, v_\theta) - \frac{in}{\epsilon_1}(g^{nj}, v_r v_\theta\bar{B}) - \frac{ij}{\epsilon_1}(g^{nj}, v_\theta v_z\bar{B})$$

$$-2\frac{\eta}{\epsilon_1}(\nu g)^{nj}_{v_r v_\theta\bar{B}} + \frac{(-1)^n}{\epsilon_1}d^j_{r\theta} + in(-1)^n d^j_{v_r^2 v_\theta\bar{B}} + ij(-1)^n d^j_{v_r v_\theta v_z\bar{B}}$$

$$+2\eta(-1)^n e^j_{v_r^2 v_\theta\bar{B}} - n^2(Qv_r^2 v_\theta\bar{B}, Q\alpha^{nj}) - j^2(Qv_\theta v_z^2\bar{B}, Q\alpha^{nj})$$

$$-2nj(v_r v_\theta v_z\bar{B}, Q\alpha^{nj}) + i\eta n(\nu\tilde{F})^{nj}_{(v_\theta^3-2v_r^2v_\theta)\bar{B}} + 4i\eta j(\nu\tilde{F})^{nj}_{v_r v_\theta v_z\bar{B}}$$

$$+2i\eta n(\nu\tilde{F})^{nj}_{v_r^2 v_\theta\bar{B}} - 2\eta(\nu'\tilde{F})^{nj}_{v_r^2 v_{\theta\bar{B}}} + 2\eta^2(\nu^2\tilde{F})^{nj}_{v_r v_\theta\bar{B}}\Big), \tag{5.4}$$

$$u_r^{nj} = \frac{i}{n^2+j^2}\left(-\frac{n}{\epsilon_1}(g^{nj},1) + \frac{-j^2\Lambda_r^{nj} + nj\Lambda_z^{nj}}{3\epsilon_1 w_1(n^2+j^2)} + n(-1)^n d_r^j + \eta n(\nu\tilde{F})^{nj}_{v_r}\right), \tag{5.5}$$

$$u_z^{nj} = \frac{i}{n^2+j^2}\left(-\frac{j}{\epsilon_1}(g^{nj},1) + \frac{nj\Lambda_r^{nj} - n^2\Lambda_z^{nj}}{3\epsilon_1 w_1(n^2+j^2)} + j(-1)^n d_r^j + \eta j(\nu\tilde{F})^{nj}_{v_r}\right). \tag{5.6}$$

Proof of Lemma 5.1: This is proved similarly to (4.10–14), only simpler. □

Lemma 5.2. *Let $\tilde{F}$ be a solution to (5.1). Then for η small enough,*

$$
\begin{aligned}
&| m_0^{00} | + | m_4^{00} | + | u_\theta^{00} | + | u_r^{00} | + | u_z^{00} | \\
&\quad \leq c\left(\frac{| g_\parallel |_2}{\epsilon_1^2} + \frac{| \tilde{\nu}^{-\frac{1}{2}} g_\perp |_2}{\epsilon_1} + | S\tilde{F} |_\sim + \frac{| \tilde{F}_b |_\sim}{\sqrt{\epsilon_1}} + \eta \parallel \tilde{F} \parallel_2\right).
\end{aligned}
$$

Proof of Lemma 5.2: Again the proof is related to the corresponding arguments in Section 4. For $(n, j) = (0, 0)$, it holds that

$$
\begin{aligned}
\alpha^{00} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} dZ\left[\frac{1}{2}(\tilde{F}(\pi - 0) + \tilde{F}(-\pi + 0)) - \sum_{n\neq 0}\alpha^{n0}e^{in\pi}\right] \\
&= \Delta - \sum_{n\neq 0}\alpha^{n0}e^{in\pi},
\end{aligned}
\tag{5.7}
$$

where $\Delta = \frac{1}{4\pi}\int_{-\pi}^{\pi} dZ(\tilde{F}(\pi - 0) + \tilde{F}(-\pi + 0))$. First,

$$
\alpha^{00}_{v_r^2\bar{A}} = \frac{1}{\sqrt{6}}\alpha_4^{00}\int v_r^2 v^2 \bar{A} M dv + \alpha^{00}_{\perp v_r^2\bar{A}}.
$$

A multiplication of (5.7) with $Mv_r^2\bar{A}$ and v-integration gives

$$
\alpha^{00}_{v_r^2\bar{A}} = \Delta_{v_r^2\bar{A}} - \sum_{n\neq 0}\alpha^{n0}_{v_r^2\bar{A}}(-1)^n.
$$

To proceed, take the scalar product of (5.1) with $v_r\bar{A}$ and identify the Fourier coefficients,

$$
\begin{aligned}
&(-1)^n d^j_{v_r^2\bar{A}} + in(v_r^2\bar{A}, \alpha^{nj}) + ij(v_r v_z\bar{A}, \alpha^{nj}) + \eta(\nu\tilde{F}_{v_\theta^2 - v_r^2})^{nj} \\
&\quad = \frac{1}{\epsilon_1}\big((v_r(v^2 - 5), \alpha^{nj}) + (g^{nj}, v_r\bar{A})\big).
\end{aligned}
\tag{5.8}
$$

Also take the scalar product of (5.1) with $v^2 - 5$, and identify the Fourier coefficients,

$$
\begin{aligned}
&-i(-1)^n d^j_{v_r(v^2-5)} + n(v_r(v^2 - 5), \alpha^{nj}) + j(v_z(v^2 - 5), \alpha^{nj}) \\
&\quad = -\frac{i}{\epsilon_1}\big((g^{nj}, v^2 - 5) + \epsilon_1\eta(\nu\tilde{F})^{nj}_{v_r(v^2-5)}\big).
\end{aligned}
\tag{5.9}
$$

Moreover, (5.1) writes

$$
v_r\frac{\partial}{\partial s}(\nu\tilde{F}) + v_z\frac{\partial}{\partial Z}(\nu\tilde{F}) - v_r\nu'\tilde{F} + \eta v^2 N\tilde{F} = \frac{1}{\epsilon_1}(L(\nu\tilde{F}) + \nu g),
$$

so that

$$i(nv_r + jv_z)(\nu\tilde{F})^{nj} + (-1)^n v_r e^j(v) - v_r(\nu'\tilde{F})^{nj} + \eta(\nu^2 N\tilde{F})^{nj}$$
$$= \frac{1}{\epsilon_1}(L(\nu\tilde{F})^{nj} + (\nu g)^{nj}),$$

where

$$e(Z,v) := \frac{1}{2\pi}\left((\nu\tilde{F})(\pi - 0, Z, v) - (\nu\tilde{F})(-\pi + 0, Z, v)\right) = \sum_{j\in Z} e^j(v)e^{ijZ}.$$

Taking the scalar product with $v_r\bar{A}$ leads to

$$(-1)^n e^j_{v_r^2\bar{A}} + in(v_r^2\bar{A}, (\nu\tilde{F})^{nj}) + ij(v_r v_z\bar{A}, (\nu\tilde{F})^{nj}) - (v_r^2\bar{A}, (\nu'\tilde{F})^{nj})$$
$$+ \eta(\nu^2\tilde{F}_{v_r^2 - v_\theta^2})^{nj} = \frac{1}{\epsilon_1}\left((v_r(v^2-5), (\nu\tilde{F})^{nj}) + (v_r\bar{A}, (\nu g)^{nj})\right). \quad (5.10)$$

By (5.8-10) for $n \neq 0$,

$$\alpha^{n0}_{v_r^2\bar{A}} = -\frac{1}{\epsilon_1^2 n^2} g^{n0}_{v^2-5} - \frac{i}{\epsilon_1} g^{n0}_{v_r\bar{A}} + \frac{(-1)^n}{\epsilon_1 n^2} d^0_{v_r(v^2-5)}$$
$$+ i\frac{(-1)^n}{n} d^0_{v_r^2\bar{A}} - \frac{\eta}{\epsilon_1 n^2}(\nu\tilde{F})^{n0}_{v_r(v^2-5)} + i\frac{\eta}{n}(\nu\tilde{F})^{n0}_{v_\theta^2 - v_r^2}$$
$$= -\frac{1}{\epsilon_1^2 n^2} g^{n0}_{v^2-5} - \frac{i}{\epsilon_1} g^{n0}_{v_r\bar{A}} + \frac{\eta}{n^2}(\nu g)^{n0}_{v_r\bar{A}}$$
$$+ \frac{(-1)^n}{\epsilon_1 n^2} d^0_{v_r(v^2-5)} + i\frac{(-1)^n}{n} d^0_{v_r^2\bar{A}} - \eta\frac{(-1)^n}{n^2} e^0_{v_r^2\bar{A}}$$
$$+ i\frac{\eta}{n}(\nu\tilde{F})^{n0}_{v_\theta^2 - v_r^2} - i\frac{\eta}{n}(\nu\tilde{F})^{n0}_{v_r^2\bar{A}}$$
$$+ \frac{\eta}{n^2}(\nu'\tilde{F})^{n0}_{v_r^2\bar{A}} - \frac{\eta^2}{n^2}(\nu^2\tilde{F})^{n0}_{v_r^2 - v_\theta^2}.$$

From here, using

$$d^0_{v_r(v^2-5)} + \eta(\nu\tilde{F})^{00}_{v_r(v^2-5)} = \frac{1}{\epsilon_1} g^{00}_{(v^2-5)},$$

it follows that

$$\mid m_4^{00}\mid_2 \leq c\Bigg(\frac{\mid g_0\mid_2 + \mid g_4\mid_2}{\epsilon_1^2} + \frac{\eta\mid g_r\mid_2 + \mid \tilde{\nu}^{-\frac{1}{2}} g_\perp\mid_2}{\epsilon_1} + \mid \tilde{F}_\perp\mid_2$$
$$+ \mid S\tilde{F}\mid_\sim + \mid \tilde{F}_b\mid_\sim + \eta\mid \tilde{F}_\parallel\mid\Bigg).$$

Since

$$m_0^{00} = \alpha_{r^2}^{00} - \frac{\sqrt{6}}{3} m_4^{00} - \alpha_{\perp r^2}^{00}, \quad u_\theta^{00} = \frac{1}{w_1}(\alpha^{00}_{v_r^2 v_\theta \bar{B}} - \alpha^{00}_{\perp v_r^2 v \theta \bar{B}}),$$

$$u_r^{00} = \Delta_r - \sum_{n\neq 0}(-1)^n u_r^{n0}, \quad u_z^{00} = \frac{1}{w_1}(\alpha^{00}_{v_r^2 v_z \bar{B}} - \alpha^{00}_{\perp v_r^2 v_z \bar{B}}),$$

similar inequalities can be obtained for m_4^{00}, u_θ^{00}, u_r^{00}, and u_z^{00} and the lemma follows. □

Proposition 5.3. *Let $\tilde{\nu}^{\frac{1}{2}}\beta \in \tilde{L}^\infty$ be given. Then there is $\eta_0 > 0$ such that for $\eta < \eta_0$, the solution $\tilde{F}$ in $\mathcal{W}^{2-}$ to*

$$v_r \frac{\partial \tilde{F}}{\partial s} + v_z \frac{\partial \tilde{F}}{\partial Z} + \eta \nu N \tilde{F} = \frac{\eta}{\epsilon}\left(\tilde{L}\tilde{F} + \epsilon \tilde{J}(\tilde{F}, \beta) + g\right), \tilde{F}_{/\partial\Omega^+} = \tilde{F}_b, \tag{5.11}$$

satisfies

$$\mid \tilde{\nu}^{\frac{1}{2}} \tilde{F} \mid_2 \leq c\left(\frac{\eta^2}{\epsilon^2} \mid g_\parallel \mid_2 + \frac{\eta}{\epsilon} \mid \tilde{\nu}^{-\frac{1}{2}} g_\perp \mid_2 + \sqrt{\frac{\eta}{\epsilon}} \mid \tilde{F}_b \mid_\sim\right). \tag{5.12}$$

Proof of Proposition 5.3: Consider first the case where $\beta = 0$. As in the axially homogeneous case, Green's formula and the spectral inequality for $\tilde{L}$ imply that

$$\epsilon_1 \mid S\tilde{F} \mid_\sim^2 + \mid \tilde{\nu}^{\frac{1}{2}} \tilde{F}_\perp \mid_2^2 \leq c(\mid \tilde{\nu}^{-\frac{1}{2}} g_\perp \mid_2^2 + \int (g_\parallel, \tilde{F}_\parallel) + \epsilon_1 \mid \tilde{F}_b \mid_\sim^2). \tag{5.13}$$

Then Lemmas 5.1–2, Parseval's identity, and (5.7) imply that

$$\mid \tilde{F}_\parallel \mid_2 \leq c\left(\frac{\mid g_\parallel \mid_2}{\epsilon_1^2} + \frac{\parallel \tilde{\nu}^{-\frac{1}{2}} g_\perp \parallel_2}{\epsilon_1} + \frac{\mid \tilde{F}_b \mid_\sim}{\sqrt{\epsilon_1}} + \mid \tilde{\nu}^{\frac{1}{2}} \tilde{F}_\perp \mid_2 + \eta \mid \tilde{F}_\parallel \mid_2\right).$$

And so (5.12) holds in the $\beta = 0$ case. The case $\beta \neq 0$ for g with g in the r.h.s can be considered as the case $\beta = 0$ for $g + \epsilon \tilde{J}(\tilde{F}, \beta)$ in the r.h.s. And so,

$$\mid \tilde{F}_\parallel \mid_2 \leq c\left(\frac{\mid g_\parallel \mid_2}{\epsilon_1^2} + \frac{\parallel \tilde{\nu}^{-\frac{1}{2}}(g_\perp + \epsilon \tilde{J}(\tilde{F}, \beta)) \parallel_2}{\epsilon_1} + \frac{1}{\sqrt{\epsilon_1}} \mid \tilde{F}_b \mid_\sim\right)$$

$$\leq c\left(\frac{\mid g_\parallel \mid_2}{\epsilon_1^2} + \frac{\mid \tilde{\nu}^{-\frac{1}{2}} g_\perp \mid_2}{\epsilon_1} + \frac{1}{\sqrt{\epsilon_1}} \mid \tilde{F}_b \mid_\sim + \eta \mid \tilde{\nu}^{\frac{1}{2}} \tilde{F} \mid_2 \mid \tilde{\nu}^{\frac{1}{2}} \beta \mid_\infty\right).$$

Since $\mid F_\parallel \mid_2 \simeq \mid \tilde{\nu}^{\frac{1}{2}} F_\parallel \mid_2$, the result holds for η small enough. □

We can now give a

Proof of Theorem 1.2: Given the asymptotic expansion φ of Section 4 and its bifurcation point, the aim is to prove the existence of a rest term R, so that for the parameters near the bifurcation point, there is an axially periodic solution

$$f = M(1 + \varphi + \epsilon R)$$

to (1.1–2) with $M^{-1} f \in \tilde{L}^\infty$. This corresponds to the function R being a solution of the same type to

$$DR = \frac{1}{\epsilon}\left(\tilde{L}R + 2\tilde{J}(R, \varphi) + \epsilon \tilde{J}(R, R) + l\right).$$

In Sections 3 and 4 a third order asymptotic expansion in ϵ was constructed in a δ^2-neighbourhood of the bifurcation velocity $u_{\theta Ab}$ with correct boundary values up to ϵ-order three, and so that l - the φ-part of the equation – is smooth in r, z and of order ϵ^3 in $\tilde{L}^q$. Here the bounds on the Φ^3-term may be obtained in the same way as those for Φ^2 in Section 4. Notice that Φ^j can be constructed so that $D\Phi^j = (I - P_0)D\Phi^j$, hence that $l = l_\perp$.

Let the sequences $(R^n)_{n\in I\!N}$ be defined by $R^0 = 0$, and

$$DR^{n+1} = \frac{1}{\epsilon}\left(\tilde{L}R^{n+1} + 2\sum_{j=1}^{3}\epsilon^j \tilde{J}(R^{n+1}, \Phi^j) + g^n\right), \tag{5.14}$$

$$R^{n+1}(1, v) = R_A(v), \quad v_r > 0, \quad R^{n+1}(r_B, v) = R_B(v), \quad v_r < 0. \tag{5.15}$$

In (5.14–15)

$$g^n := \epsilon^2 \tilde{J}(R^n, R^n) + l,$$

$$\epsilon R_A(v) := e^{\epsilon u_{\theta A1} v_\theta - \frac{\epsilon^2}{2} u_{\theta A1}^2} - 1 - \sum_{j=1}^{3} \epsilon^j \Phi^j(r_A, v), \quad v_r > 0,$$

$$\epsilon R_B(v) := 0, \quad v_r < 0,$$

with $R_b = (R_A, R_B)$ of ϵ-order three. □

For the rest term iteration scheme (5.14–15) the following holds.

Proposition 5.4. *For $\epsilon > 0$ and small enough together with $\eta = r_B - r_A$, there is a unique sequence (R^n) of solutions to (5.14-15) in the set $X := \{R; \mid \tilde{\nu}^{\frac{1}{2}} R \mid_q \leq K\}$ for some constant K. The sequence converges in $\tilde{L}^q$ for $2 \leq q \leq \infty$, to an isolated solution of*

$$DR = \frac{1}{\epsilon}\left(\tilde{L}R + \epsilon\tilde{J}(R, R) + 2\tilde{J}(R, \varphi) + l\right), \tag{5.16}$$

$$R(1, v) = R_A(v), \quad v_r > 0, \quad R(r_B, v) = R_B(v), \quad v_r < 0. \tag{5.17}$$

When ϵ tends to zero, the corresponding hydrodynamic moments converge to solutions of the limiting fluid equations at the leading order ϵ.

Proof of Proposition 5.4: The existence result of Proposition 2.2 holds for the boundary value problem

$$Df = \frac{1}{\epsilon}\left(\tilde{L}f + 2\sum_{j=1}^{3}\epsilon^j \tilde{J}(f, \Phi^j) + g\right),$$

$$f(1, v) = R_A(v), \quad v_r > 0, \quad f(r_B, v) = R_B(v), \quad v_r < 0.$$

Rescale in space to $(-\pi, \pi)^2$ and consider the approximation (5.14-15) in the case $n = 0$ with $g^0 = l$. As discussed before (5.14), this $g^0 = g^0_\perp$ is of order ϵ^3 in $\tilde{L}^\infty$, and

$$|\ \tilde{\nu}^{-\frac{1}{2}} l\ |_\infty + |\ R_b\ |_\sim \leq c_1\epsilon^3,$$

for some constant c_1. By (5.12) and (2.20) it holds that for some constant c_2

$$|\ \tilde{\nu}^{\frac{1}{2}} R^1\ |_2 \leq c_1 c_2 \eta\epsilon^2, \quad |\ \tilde{\nu}^{\frac{1}{2}} R^1\ |_\infty \leq 2c_1c_2\eta\epsilon, \tag{5.18}$$

for η and ϵ small enough. Let us prove by induction that

$$\begin{aligned} &|\ \tilde{\nu}^{\frac{1}{2}} R^n\ |_\infty \leq 4c_1c_2\epsilon, \\ &\quad |\ \tilde{\nu}^{\frac{1}{2}}(R^{n+1} - R^n)\ |_2 \leq 2c_1c_2\epsilon\ |\ \tilde{\nu}^{\frac{1}{2}}(R^n - R^{n-1})\ |_2, \quad n \geq 1. \end{aligned} \tag{5.19}$$

For $n = 1$, $R^2 - R^1$ satisfies

$$D(R^2 - R^1) = \frac{\eta}{\epsilon}\left(\tilde{L}(R^2 - R^1) + 2\sum_{j=1}^{3}\epsilon^j\tilde{J}(R^2 - R^1, \Phi^j) + \epsilon\tilde{J}(R^1, R^1)\right),$$

$$(R^2 - R^1)(r_A, z, v) = 0, \quad v_r > 0, \quad (R^2 - R^1)(r_B, z, v) = 0, \quad v_r < 0,$$

so that, by (5.12),

$$|\ \tilde{\nu}^{\frac{1}{2}}(R^2 - R^1)\ |_2 \leq c_2\eta\ |\ \tilde{\nu}^{-\frac{1}{2}}\tilde{J}(R^1, R^1)\ |_2\ .$$

Recall that for any $g \in \tilde{L}^\infty$ resp. $h \in \tilde{L}^q$,

$$|\ \tilde{\nu}^{-\frac{1}{2}}\tilde{J}(g, h)\ |_q \leq c_3\ |\ \tilde{\nu}^{\frac{1}{2}} g\ |_\infty |\ \tilde{\nu}^{\frac{1}{2}} h\ |_q\ . \tag{5.20}$$

Hence

$$|\ \tilde{\nu}^{\frac{1}{2}}(R^2 - R^1)\ |_2 \leq c_1\eta^2\epsilon\ |\ \tilde{\nu}^{\frac{1}{2}}(R^1 - R^0)\ |_2,$$

for η small enough. If (5.19) holds until n, then

$$|\ \tilde{\nu}^{\frac{1}{2}} R^{n+1}\ |_\infty \leq |\ \tilde{\nu}^{\frac{1}{2}}(R^{n+1} - R^n)\ |_\infty + \cdots + |\ \tilde{\nu}^{\frac{1}{2}}(R^1 - R^0)\ |_\infty$$

$$\leq \frac{c_4}{\epsilon}(|\ \tilde{\nu}^{\frac{1}{2}}(R^{n+1} - R^n)\ |_2 + \cdots + |\ \tilde{\nu}^{\frac{1}{2}}(R^1 - R^0)\ |_2)$$
$$\leq 4c_1c_2\epsilon,$$

for η small enough. Then $R^{n+2} - R^{n+1}$ satisfies

$$D(R^{n+2} - R^{n+1}) = \frac{1}{\epsilon}\left(\tilde{L}(R^{n+2} - R^{n+1}) + 2\sum_{j=1}^{3}\epsilon^j\tilde{J}(R^{n+2} - R^{n+1}, \Phi^j) + \epsilon\tilde{J}(R^{n+1} + R^n, R^{n+1} - R^n)\right)$$

$$(R^{n+2} - R^{n+1})(r_A, z, v) = 0, \quad v_r > 0,$$
$$(R^{n+2} - R^{n+1})(r_B, z, v) = 0, \quad v_r < 0,$$

so that by (5.12) and the bound on $|\ \tilde{\nu}^{\frac{1}{2}}R^n\ |_\infty$ and $|\ \tilde{\nu}^{\frac{1}{2}}R^{n+1}\ |_\infty$,

$$|\ \tilde{\nu}^{\frac{1}{2}}(R^{n+2} - R^{n+1})\ |_2 \leq c_3\eta(|\ \tilde{\nu}^{\frac{1}{2}}R^{n+1}\ |_\infty + |\ \tilde{\nu}^{\frac{1}{2}}R^n\ |_\infty)\ |\ \tilde{\nu}^{\frac{1}{2}}(R^{n+1} - R^n)\ |_2$$
$$\leq 2c_1c_2\epsilon\ |\ \tilde{\nu}^{\frac{1}{2}}(R^{n+1} - R^n)\ |_2,$$

for ϵ and η small enough.

And so (R^n) converges for sufficiently small $\eta > 0$ to some R, solution to (5.16–17) in $\tilde{L}^q$ for $q \leq \infty$. The contraction mapping construction guarantees that this solution is isolated.

It finally follows from the above proof that, when ϵ tends to zero, the hydrodynamic moments converge to the (Hilbert type) solutions of the corresponding leading (first) order limiting fluid Taylor Couette equations. □

End of proof of Theorem 1.2: The theorem is now an immediate consequence of Proposition 5.4. □

6. POSITIVITY

Write $f = f^+ - f^-$ with $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. This section looks into the positivity of the isolated solutions to (1.1–2) obtained in the previous sections. Suppose f satisfies the related problem (6.1–2) below. Then $f^- = 0$ by Theorem 6.1, and $f = f^+$ is a non-negative solution also to (1.1–2). If the contraction mapping approach used above could be extended to the construction of suitable solutions for the problem (6.1–2), then as a consequence, any solution from the previous sections would coincide with such a non-negative solution.

Theorem 6.1. *Let Ω be a bounded set in $\mathbb{R}^n$ with smooth boundary, and f_b a nonnegative function defined on $\partial\Omega^+$. If $M^{-1}f \in \tilde{L}^\infty(\Omega \times I\!R^3)$ and f solves the boundary value problem*

$$v \cdot \nabla_x f = Q(f^+, f^+) - M\tilde{L}(M^{-1}f^-), \quad (x, v) \in \Omega \times \mathbb{R}^3, \tag{6.1}$$

$$f = f_b, \quad \partial\Omega^+, \tag{6.2}$$

then $f^- = 0$, and $f = f^+$ solves the corresponding boundary value problem for the Boltzmann equation,

$$v \cdot \nabla_x f = Q(f, f), \quad \Omega \times \mathbb{R}^3,$$

$$f = f_b, \quad \partial\Omega^+.$$

Proof of Theorem 6.1: The function $F = M^{-1}f$ satisfies

$$v \cdot \nabla_x F = \tilde{J}(F^+, F^+) - \tilde{L}(F^-), \quad F = M^{-1}f_b, \quad \partial\Omega^+.$$

Define $\tilde{J}^+$ and $\tilde{J}^-$ by $\tilde{J}(\varphi, \varphi) = \tilde{J}^+(\varphi, \varphi) - \tilde{J}^-(\varphi, \varphi)$, where

$$\tilde{J}^+(\varphi, \varphi)(v) := \int | v - v_* |^\beta \, b(\theta) M_* \varphi' \varphi'_* dv_* d\omega,$$

$$\tilde{J}^-(\varphi, \varphi)(v) := \varphi(v) \int | v - v_* |^\beta \, b(\theta) M_* \varphi_* dv_* d\omega.$$

Also, F^- satisfies

$$-v \cdot \nabla_x F^- = \chi_{F^- \neq 0}(\tilde{J}^+(F^+, F^+) - \tilde{L}(F^-)), \tag{6.3}$$

$$F^- = 0, \quad \partial\Omega^+.$$

Multiplying (6.3) with $-MF^-$, integrating on $\Omega \times \mathbb{R}^3$ and using that

$$-\int MF^- \chi_{F^- \neq 0} \tilde{L}(F^-) dv = -\int MF^- \tilde{L}(F^-) dv$$

$$\geq c \int M\tilde{\nu} \, | (I - P_0)F^- |^2 \, dv,$$

implies that

$$\frac{1}{2} \int_{\partial\Omega^-} | v \cdot n | \, M(F^-)^2 + c \int_{\Omega \times I\!R^3} M\tilde{\nu} \, | (I - P_0)F^- |^2$$

$$\leq -\int MF^- \chi_{F^- \neq 0} \tilde{J}^+(F^+, F^+) \leq 0.$$

It follows that

$$F^- = 0 \text{ on } \partial\Omega^-, \quad \tilde{L}(F^-) = 0.$$

And so, F^- satisfies

$$F^- = 0, \quad \partial\Omega^- \cup \partial\Omega^+, \quad v \cdot \nabla_x F^- \leq 0.$$

This implies that F^- is identically zero. □

Corollary 6.1. *If there is a solution f to (6.1–2) in the ball of contraction of the proofs in Section 2 or Section 5, then $f^- = 0$ and $f = f^+$ is the unique and strictly positive solution in that ball of the boundary value problem (1.1–2).*

Proof of Theorem 1.3: We shall end by a discussion of Maxwellian molecules, for which there is indeed a solution to (6.1–2), i.e. the hypothesis of the corollary holds, and start with the axially homogeneous situation of Section 2. Set $\bar{\chi} = \chi_{|v|<\epsilon^{-\frac{1}{n}}}$ and denote again by φ the asymptotic expansion of order two,

$$\varphi(r, v) = \sum_{i=1}^{2} \epsilon^i \Phi^i.$$

In the frame of this paper, if the terms in Φ^i, $1 \leq i \leq 2$ are polynomially bounded in the v-variable, with bounded coefficients in the r-variable, then for ϵ and $\frac{1}{n}$ small enough and positive, it would hold that

$$1 + \bar{\chi}\Phi = 1 + \bar{\chi}\left(\sum_{i=1}^{2} \epsilon^i \Phi^i\right) \geq 0. \tag{6.4}$$

The required bounds follow from the previous discussion of the terms in φ except $\bar{A}$, $\bar{B}$-terms in Φ^2 and Φ^3. But it is well known that also the $\bar{A}$ and $\bar{B}$ terms are polynomially bounded in the Maxwellian case (see ref.[(3)]) Notice that the $\tilde{L}^q$-norm of $(1 - \bar{\chi})\Phi$ for any q is of arbitrarily high order in ϵ.

Using the approach of Section 2, the positivity under the cut-off $\bar{\chi}$ in (6.4), and the corresponding splitting

$$f = M(1 + \bar{\chi}\varphi + \epsilon R),$$

lead to a nonnegative solution of (1.1–2) with $M^{-1} f \in \tilde{L}^\infty$ as follows. Namely, the rest term R should be a solution to

$$DR = \frac{1}{\epsilon}\left(\tilde{L}R + 2\tilde{J}(\bar{R}, \bar{\chi}\varphi) + \epsilon\tilde{J}(\bar{R}, \bar{R}) + \bar{l}\right), \tag{6.5}$$

where

$$\bar{l} = \frac{1}{\epsilon}\left(\tilde{L}(\bar{\chi}\varphi) + \tilde{J}(\bar{\chi}\varphi, \bar{\chi}\varphi) - \epsilon D(\bar{\chi}\varphi)\right),$$

and

$$\bar{R}(r,v) = R(r,v) \quad \text{when} \quad \epsilon R(r,v) \geq -\left(1+\bar{\chi}\sum_{i=1}^{2}\epsilon^i \Phi^i(r,v)\right),$$

$$\bar{R}(r,v) = -\frac{1}{\epsilon}\left(1+\bar{\chi}\sum_{i=1}^{2}\epsilon^i \Phi^i(r,v)\right) \quad \text{otherwise.}$$

Here $\bar{l}$ can be decomposed as $\bar{l}_\perp$with the same properties as l in Section 2, and $\bar{l}_\parallel$ which in $\tilde{L}^q$ is of arbitrarily high order in ϵ. The approximating sequences $(R^n)_{n\in\mathbb{N}}$ and $(\bar{R}^n)_{n\in\mathbb{N}}$ are defined by $R^0 = \bar{R}^0 = 0$, and

$$DR^{n+1} = \frac{1}{\epsilon}\left(\tilde{L}R^{n+1} + 2\sum_{j=1}^{2}\epsilon^j \tilde{J}(\bar{R}^{n+1}, \bar{\chi}\Phi^j) + g^n\right), \tag{6.6}$$

$$R^{n+1}(1,v) = R_A(v), \quad v_r > 0, \quad R^{n+1}(r_B,v) = R_B(v), \quad v_r < 0, \tag{6.7}$$

with

$$g^n := \epsilon\tilde{J}(\bar{R}^n, \bar{R}^n) + \bar{l},$$

$$\epsilon R_A(v) := e^{\epsilon u_{\theta A1}v_\theta - \frac{\epsilon^2}{2}u_{\theta A1}^2 v_\theta^2} - 1 - \bar{\chi}\Phi(r_A, v), \quad v_r > 0,$$

$$\epsilon R_B(v) := -\bar{\chi}\Phi(r_B, v), \quad v_r < 0,$$

and

$$\bar{R}^n(r,v) = R^n(r,v) \quad \text{when} \quad \epsilon R^n(r,v) \geq -\left(1+\bar{\chi}\sum_{i=1}^{2}\epsilon^i \Phi^i(r,v)\right),$$

$$\bar{R}^n(r,v) = -\frac{1}{\epsilon}\left(1+\bar{\chi}\sum_{i=1}^{2}\epsilon^i \Phi^i(r,v)\right) \quad \text{otherwise.}$$

From here the only difference with respect to the contraction mapping analysis of Section 2, is related to the appearance of a term $g_\parallel^0$ of arbitrarily high order, and of factors $\bar{R}^n$ instead of the previous R^n in $\tilde{J}$. Proposition 2.2 is not changed by the replacements $\bar{R}$. Arguing similarly to the previous cases, the contribution to the a priori estimate (2.19) due to $g_\parallel$ gives rise to an extra term $\mid g_\parallel \mid_2 \epsilon^{-1}$, hence

$$\begin{aligned}\epsilon^{\frac{1}{2}} \mid \mathcal{S}F \mid_\sim + \mid \tilde{\nu}^{\frac{1}{2}} F_\perp \mid_2 \leq c(&\mid \tilde{\nu}^{-\frac{1}{2}} g_\perp \mid_2 + \epsilon^{-1} \mid \tilde{\nu}^{-\frac{1}{2}} g_\parallel \mid_2 + \epsilon \mid F_\parallel \mid_2 \\ &+ \epsilon^{\frac{1}{2}} \mid F_b \mid_\sim).\end{aligned} \tag{6.8}$$

For the hydrodynamic part the proof of Proposition 3.1 from ref.[1] is essentially unchanged in the present situation (with the $\bar{R}$-terms included in $g_\perp$ and without the here unnecessary switch to R_-). The hydrodynamic estimate (2.23) follows.

We turn to the existence proof for (6.5), (6.7). In the new situation the contraction mapping arguments from the proof of Proposition 2.5 still hold. That leads to an isolated solution for (6.5), (6.7) which defines the positive solution of Corollary 6.2. The solution lies in the same ball of contraction as the solution constructed in Section 2, so they coincide and the solution of Section 2 is positive. That completes the proof of Theorem 1.3 in the axially homogeneous case. The axially symmetric case for Maxwellian molecules is similarly proved. □

Remark. This positivity analysis with some further technical steps added, also holds for Maxwellian molecules and the solutions obtained in ref.[1] The only obstacle for extending the above approach to hard forces is a lack of growth estimates at zero and infinity for certain terms in the asymptotic expansion φ, like the terms $v_r \bar{A}$ and $v_\theta v_r \bar{B}$.

ACKNOWLEDGMENT

The authors thank Professor Y. Sone and Professor K. Aoki for useful discussions about their asymptotic techniques and results. The research was carried out within the program of the HYKE network under European Union contract HPRN-CT-2002-00282.

REFERENCES

1. L. Arkeryd and A. Nouri, The stationary nonlinear Boltzmann equation in a Couette setting with multiple, isolated L^q-solutions and hydrodynamic limits, *Journ. Stat. Phys.* **118**:849–881 (2005).
2. C. Bardos, E. Caflish, and B. Nicolaenko, The Milne and Kramers problems for the Boltzmann equation of a hard sphere gas, *Comm. Pure Appl. Math.* **39**:323–352 (1986).
3. C. Cercignani, *Mathematical methods in kinetic theory*, Plenum Press New York, 1990.
4. P. Chossat and G. Iooss, *The Couette-Taylor problem*, Springer-Verlag Berlin, 1994.
5. W. A. Coppel, *Disconjugacy*, Springer-Verlag, 1971.
6. L. Desvillettes, Sur quelques hypothèses nécessaires à l'obtention du développement de Chapman-Enskog, *Preprint*, 1994.
7. F. Golse and F. Poupaud, Stationary solutions of the linearized Boltzmann equation in a half space, *Math. Meth. Appl. Sci.* **11**:483–502 (1989).
8. H. Grad, Asymptotic theory of the Boltzmann equation I, *Phys. Fluids* **6**:147–181 (1963).
9. H. Grad, Asymptotic theory of the Boltzmann equation II, *Rarefied Gas Dynamics*, pp. 26–59, Acad. Press NY, 1963.
10. J. P. Guiraud, Problème aux limites intérieur pour l'équation de Boltzmann en régime stationnaire, faiblement non linéaire, *J. Mécanique* **11**:183–231 (1972).
11. M. N. Kogan, *Rarefied Gas Dynamics*, Plenum NY, 1969.

12. N. Maslova, *Nonlinear evolution equations, Kinetic approach*, Series on Advances in Mathematics for Applied Sciences, Vol. 10, World Scientific, 1993.
13. Y. Sone, *Kinetic Theory and Fluid Dynamics*, Birkhäuser Boston, 2002.
14. Y. Sone and T. Doi, Bifurcation of a flow of a gas between rotating coaxial circular cylinders with evaporation and condensation, *Rarefied Gas Dynamics*, AIAA New York, 2002, 646–653.
15. W. Velte, Stabiliät und Verzweigung stationärer Lösungen der Navier-Stokesschen Gleichungen beim Taylorproblem, *Arch. Rat. Mechs. Anal.* **22**:1–14 (1966).